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AN INADEQUACY IN KRIPKE-SEMANTICS FOR INTUITIONISTIC QUANTIFICATIONAL LOGIC

The semantics for intuitionistic quantificational logic, that have come to be known as Kripke-semantics (see, e.g., [2], p. 246) after the influential presentation of Kripke [1], turn out to be unsound. Since a large body of theory concerning intuitionistic logics and mathematics is now based on these semantics the matter is of more than merely local significance.

Although the points made apply equally against many other presentations of Kripke-semantics for intuitionist logic (e.g. those of Thomason [3], Aczel [4], Gabbay [5] and elsewhere), it is convenient to focus on Kripke [1], and to borrow his terminilogy and notation. Kripke in turn adopts (see [1], p. 93) the formulation of intuitionistic predicate logic of Kleene [6], and it is advantageous to follow suit. It is worth noting that Kleene's formulation allows both for free variables and for constants (and so also do the formulations adopted by some others: e.g. Thomason [3], p. 1 and Aczel [4], p. 2).

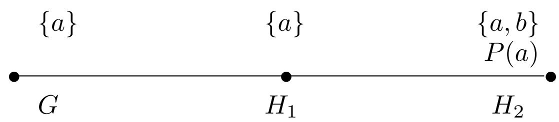
The Kripke-semantics fails to ensure soundness for intuitionistic theses of the form

(1) $A(y) \supset (\exists x)A(x)$.

That (1) is a theorem scheme of intuitionistic predicate logic is immediate from Kleene's axiomatization ([6], p. 82): it is the free-variable case of his axiom scheme A11. But not all instances of (1) are sound; in particular soundness arguments are liable to break where $A(x)$ is a negated or implicational formula, i.e. of the forms $\neg B(x)$ or $B(x) \supset C(x)$. Consider, for simplicity, the formula

(2) $\neg P(y) \supset (\exists x)\neg P(x)$, where P is a one-place predicate letter and y is a free variable (or constant).

As in Kripke ([1], p. 101) countermodel details can be neatly encapsulated in a diagram:



Thus $\phi(P(x), H_2) = T$ when x is assigned a, but $\phi(P(y), H_2) = \phi(P(y), H_1) = F$, with y assigned b, as will be assumed. More explicitly, the quantificational model structure $M = (G, K, R, \psi)$ used is defined as follows: $K = \{G, H_1, H_2\}$; $R = \{\langle G, H_1\rangle, \langle H_1, H_2\rangle, \langle G, H_2\rangle, \langle G, G\rangle, \langle H_1, H_1\rangle, \langle H_2, H_2\rangle\}$; $\psi(G) = \psi(H_1) = \{a\}$, $\psi(H_2) = \{a, b\} = U$. For the quantificational model ϕ on M it is required that $\phi(P, H_2) = \{a\}$; otherwise ϕ can be determined arbitrarily in accord with Kripke's prescription (p. 95). Note that $b \notin \phi(P, H_1)$, since $b \notin \psi(H_1)$. (If the alternative Kripke mentions on p. 96, of allowing $\phi(P, H)$ to be any subset of U, were adopted, then one would require $\phi(P, H_1) = \{a\}$ or $\{\ \}$.)

Model ϕ is a countermodel to (2), for assignment $b \in U$ to free variable y, as the following details establish. As $b \notin \phi(P, H_1)$, $b \notin \phi(P, H_2)$, $\phi(P(y), H_1) = \phi(P(y), H_2) = F$. Thus for every $H \in K$ such that $H_1 R H$, $\phi(P(y), H) = F$; so $\phi(\neg P(y), H_1) = T$. When x is assigned element a of U, $\phi(P(x), H_2) = T$, as $a \in \phi(P, H_2)$. Thus since $H_1 R H_2$ and $\phi(P(x), H_2) = T$, $\phi(\neg P(x), H_1) = F$. Hence for every assignment of an element of $\psi(H_1)$ to x, $\phi(\neg P(x), H_1) = F$; so $\phi((\exists x)\neg P(x), H_1) = F$. Finally, as $G R H_1$, by the rule for connective $\supset$, $\phi((2), G) = F$. (Observe that the three world model structure includes a two world model structure $\{H_1, H_2\}$ with base H_1 which also falsifies (2).)

A formula A of intuitionistic quantification theory is supposed to be valid iff $\phi(A, G) = T$ for every model ϕ on every *q.m.s.* for every assignment of elements of U to the free individual variables (and constants) of A. (This account of validity, not given explicitly in [1], may however be extracted from remarks in [1], pp. 95, 96 and especially 120–1. It is explicit elsewhere in Kripke's work, e.g. [7], p. 86 ff, and also in later papers based on [1], e.g. Thomason [3], p. 4.) It should follow from the countermodel given that (2) is not valid.

The precisely analogous problems arise with semantic tableaux is shown by the following tableau countermodel for (2):

$t_0; \psi(t_0) = \{x\}$		$\neg P(y) \supset (\exists x)\neg P(x)$	Start
	$\downarrow S$		
$t_1; \psi(t_1) = \{x\}$	$\neg P(y)$	$(\exists x)\neg P(x)$	by Pr
		$P(y)$	by Nl
		$\neg P(x)$	by Σr, as $\psi(t_1) = \{x\}$
	$\downarrow S$		
t_2	$P(x)$		by Nr
	$\neg P(y)$		by hereditary requirement
		$P(y)$	by Nl

The construction terminates but the tableau is not closed.

The problem with existential generalisation (1) resembles, of course, the feature of free logic that the variable particularised upon may not belong to the restricted domain of the existential quantifier; and at least as far as *pure* intuitionistic quantification logic is concerned, the problem may be rectified after the fashion in which free logic is modelled classically (cf. Kripke [7]), namely by redefining validity, thus:

A is *valid* iff $(A^c, G) = T$ for every model ϕ in every *q.m.s.*, where A^c is the universal closure of A.

The problem is circumvented in this fashion, as the semantic tableau for $(y)(\neg P(y) \supset (\exists x)\neg P(x))$ does close:

$t_0; \psi(t_0) = \{x\}$	$(y)(\neg P(y) \supset (\exists x)\neg P(x))$	
$\downarrow S$		
$t_1; \psi(t_1) = \{x, y\}$	$\neg P(y) \supset (\exists x)\neg P(x)$	by Πr
$\downarrow S$		
$t_2; \psi(t_2) = \{x, y\}\neg P(y)$	$(\exists x)\neg P(x)$	by Pr
	$\neg P(x)$	by Σr
	$\neg P(y)$	by Σr

This resolution of the problem soon suggests somewhat improved resolutions; in particular, admit formulae with constants and free individual variables but require in the definition of validity that these are assigned elements of $\psi(G)$, not just elements of U, which may not yet have been introduced. An equivalent restriction is adopted in Schütte ([8], p. 46), as A. Raggio in effect pointed out to me upon reading an earlier version of this note. Where validity is determined at G, as in Kripke's work, the restriction comes to this: A model ϕ on M is *admissible for* formula A iff the assignments to free variables (and constants) appearing in A all belong to $\psi(G)$, and a formula A is *valid* iff $I(A, G) = T$ for every admissible model for A. Essentially the same restriction on the definition of validity is imposed by Fitting ([9], p. 46). Yet both Schütte and Fitting claim to be presenting Kripke's semantics (thus, according to Fitting, his 'modelling structure is due to Kripke and may be found, in different notation, in' [1]), and nowhere remark upon its inadequacy.

But while the resolutions do work, and suffice to reinstate many of the applications that have been made of Kripke-semantics (e.g. those of Gabbay in [5] and elsewhere), they are hardly satisfactory. For an intuitionist may very well want to regard new constants introduced in later evidential situations (and not all there to begin with) as admissible values of free variables. To encompass such real cases the Kripke-semantics will have, it seems, to be complicated in a way not admitted by the formalistic resolution outlined.

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