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RESOLUTION SYSTEM FOR ω^+-VALUED LOGIC

The ω^+-valued Post logic was introduced by Rasiowa in [3]. The applications of this logic are connected with the formalized theory of programs and for this reason we are interested in developing a theorem proving system for it. The idea of a resolution system follows from [5], where such a system for the classical predicate calculus was presented.

1. The ω^+-valued Post logic

The formalized language L of the ω^+-valued Post logic is obtained from the language of the intuitionistic predicate calculus by adding the symbols D_i, $1 \leqslant i < \omega$, of unary propositional connectives and the symbols E_i, $0 \leqslant i < \omega$, of propositional constants. The semantics of the language L is given by means of the notion of realization of formulas and terms in a non-empty set and in the Post algebra $\mathcal{P}_\omega$ of order ω^+, which is defined as follows: $\mathcal{P}_\omega = (P_\omega, e_\omega, \cup, \cap, \supset, -, (d_i)_{1 \leqslant i < \omega}, (e_i)_{0 \leqslant i \leqslant \omega})$, $P_\omega = \{e_i\}_{0 \leqslant i \leqslant \omega}$ is an enumerable chain with $e_i \leqslant e_j$ for $i \leqslant j$ and $e_j \neq e_j$ for $i \neq j$, e_0, e are the smallest and the greatest element in P_ω respectively, the algebra $(P_\omega, e_\omega, \cup, \cap, \supset, -)$ is the complete pseudo-Boolean algebra ([4], p. 58) such that e_ω is the unit element, $e_i \cup e_j = e_{max(i,j)}, e_i \cap e_j = e_{min(i,j)}, e_i \supset e_j = e_\omega$ for $i \leqslant j$, $e_i \supset e_j = e_j$ for $i > j$, $-e_i = e_i \supset e_0$, the unary operations d_i are defined by the equations $d_i(e_j) = e_\omega$ for $i \leqslant j$, $d_i(e_j) = e_0$ for $i > j$. The realizations of the language L is a non-empty set J and in the algebra $\mathcal{P}_\omega$ are defined as usual ([3]), and with respect to the added connectives satisfy the following conditions: $(D_i\alpha)_R(v) = d_i\alpha_R(v)$, $E_{iR} = e_i$ for any valuation v of individual variables in the set J. A formula α of the language L is a tautology if $\alpha_R(v) = e_\omega$ for every realization R and for every valuation v.

A realization R is said to be a model of a formula α if $\alpha_R(v) = e_\omega$ for every valuation v. We admit the usual definitions of an elementary formula, a closed formula, a subformula of a formula, a free variable.

The following theorems are easily obtained by using the results of [3].

(1.1) For any formula $D_i\alpha$ there exists a formula in prenex form equivalent to it.

(1.2) For any open formula of the form $D_i\alpha$ there is a normal form formula equivalent to it, being a conjunction of disjunctions of formulas of the form $D_j\beta$ or $\neg D_j\beta$ for an elementary formula β.

(1.3) For any formula α the following conditions are equivalent
- (a) α is a tautology,
- (b) for every i the formula $D_i\alpha$ is a tautology,
- (c) for every i the formula $\neg D_i\alpha$ is unsatisfiable.

(1.4) For any formula α the following conditions are equivalent
- (a) α is unsatisfiable,
- (b) the set $\{D_i\alpha\}_{1\leqslant i<\omega}$ is unsatisfiable.

Given a formula α, and applying the Skolem method of elimination of quantifiers to all formulas $D_i\alpha$, $1 \leqslant i < \omega$, we obtain the set S_α of formulas in Skolem form such that the following theorem holds.

(1.5) For any formula α the following conditions are equivalent
- (a) α has a model,
- (b) S_α has a model.

2. Resolution and factoring

Let I be the set consisting of all formulas of the form $D_i\alpha$ or $\neg D_i\alpha$ for an elementary formula α. Let $\mathbf{\Gamma}_I$ be the family of all sets of the form $\{D_i\alpha, \neg D_j\alpha\}$ for $j \leqslant i$ and for an elementary formula α. Let us adjoin to the language L the propositional constant $\Box$ such that for any realization R $\Box_R \neq e_\omega$. Let C_I be the least set containing $I \cup \{\Box\}$ and such that if $\alpha, \beta \in C_I$ then $\alpha \vee \beta \in C_I$. The elements of the set C_I will be called clauses and they will be identified with the sets of their subformulas from the set I. By a substitution we mean any mapping from the set V of individual variables of the language L into the set T of terms of the language L, extended in the usual way onto the set F of all formulas. We will say that a substitution s reduces a set $\{\mu_1, \ldots, \mu_n\} \subseteq I$, $1 \leqslant n < \omega$, to a set $\{\nu_1, \ldots, \nu_n\} \subseteq I$ if $s\mu_i = \nu_i$ for each $i = 1, \ldots, n$. We will say that a substitution s unifies a set $\{\mu_1, \ldots, \mu_n\}$ if $s\mu_i = s\mu_j$ for each $i, j = 1, \ldots, n$. A substitution s reducing a set $A \subseteq I$ to a set $B \subseteq I$ is called a least reducing substitution if for any substitution s_1 which reduces A to B there is a substitution s_2 such that $s_1 = ss_2$. In the same way we define the least unifying substitution for a set A. We will say that a set A of formulas is reducible to a set B of formulas if there is a substitution which reduces A to B, and we will say that a set A is unifiable if there is a substitution which unifies A. The algorithm for finding a least unifying substitution for a unifiable set of formulas is given in [5]. By a slight modification of this algorithm we obtain the algorithm for finding a least reducing substitution for a set of formulas reducible to another set of formulas.

For any set $\Gamma \in \mathbf{\Gamma}_I$ let $r_\Gamma \subseteq C_I^3$ be the relation defined as follows:

(r) $r_\Gamma(\alpha_1, \alpha_2, \alpha)$ iff clauses α_1, α_2 have no free variables in common, for each clause α_1, α_2 there is its subformula

$\mu_1, \mu_2 \in I$, respectively, and there is a least substitutions reducing the set $\{\mu_1, \mu_2\}$ to a certain $\Gamma \in \mathbf{\Gamma}_I$ and $\alpha = s((\alpha_1 - \{\mu_1\}) \vee (\alpha_2 - \{\mu_2\}))$.

Every r_Γ will be called a resolution rule. Let $f \subseteq C_I^2$ be the relation defined as follows:

(f) $f(\alpha, \beta)$ iff there exist subformulas $\mu_1, \ldots, \mu_n \in I$, $1 \leqslant n < \omega$, of α and there is a least substitution unifying the set $\{\mu_1, \ldots, \mu_n\}$ and $\beta = s(\alpha - \{\mu_2, \ldots, \mu_n\})$.

The relation f is called the factoring rule.

We define the operations R and F in the powerset $P(C_I)$ in the following way:

(R) $R(A) = \{\alpha \in C_I : (\exists \Gamma \in \mathbf{\Gamma}_I)(\exists \alpha_1, \alpha_2 \in A) r_\Gamma(\alpha_1, \alpha_2, \alpha)$,
(F) $F(A) = \{\beta \in C_I : (\exists \alpha \in A) f(\alpha, \beta)\} \cup A$.

(2.1) For any realization R the following conditions are satisfied

(a) if $f(\alpha, \beta)$ holds and if $\alpha_R(v) = e_\omega$ for every valuation v then $\beta_R(v) = e_\omega$ for every valuation v,
(b) if $r_\Gamma(\alpha_1, \alpha_2, \alpha)$ holds and $\alpha_{1R}(v) = e_\omega = \alpha_{2R}(v)$ for every valuation v, then $\alpha_R(v) = e_\omega$ for every valuation v.

3. Resolution system

Let us consider the family $\{D^n\}_{0 \leqslant n < \omega}$ of operations and the operation D in the powerset $P(C_I)$, defined as follows:

$(D1)$ $D^0(A) = A$, for $n > D^n(A) = D^{n-1}(A) \cup R(F(D^{n-1}(A)))$,
$(D2)$ $D(A) = \bigcup\limits_{0 \leqslant n < \omega} D^n(A)$.

The system (C_I, D) will be called the resolution system. The following theorems hold.

(3.1) For any sets A, B of clauses the following conditions are satisfied
(a) $A \subseteq D(A)$,
(b) if $A \subseteq B$ then $D(A) \subseteq D(B)$,
(c) $D(D(A)) = D(A)$.
(3.2) For any set A of clauses if $\alpha \in D(A)$, then every model of A is the model of α.
(3.3) For any set A of clauses if $\alpha \in D(A)$, then there exists a finite subset B of the set A such that $\alpha \in D(B)$.

4. Completeness

The ordinary completeness theorem does not hold for the system (C_I, D). Th following lemma gives the so called resolution-completeness of the system (C_I, D).

(4.1) For any set A of clauses the following conditions are equivalent
(a) A has no model,
(b) $\Box \in D(A)$.

Given a formula α of the language L, let $A_{D_i\alpha}(A_{\neg D_i\alpha})$ denote the set of all disjunctions occurring in a normal form formula of the Skolem form formula of $D_i\alpha(\neg D_i\alpha)$. The following lemmas show how the presented resolution system can be used to prove theorems of the ω^+-valued logic.

(4.2) For any closed formula α the following conditions are equivalent
(a) α is a tautology,
(b) for every i, $1 \leqslant i < \omega$, the set $A_{\neg D_i\alpha}$ has no model,
(c) for every i $\Box \in D(A_{\neg D_i\alpha})$.

(4.3) For any closed formula α the following conditions are equivalent
(a) α is unsatisfiable,
(b) the set $\bigcup_{1\leqslant i<\omega} A_{D_i\alpha}$ has no model,
(c) $\Box \in D(\bigcup_{1\leqslant i<\omega} A_{D_i\alpha})$.

The proofs of all theorems presented in this paper are given in [1] and [2].

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