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BOOLEAN THEORIES WITH QUANTIFIERS*

§1. In this paper we are concerned with some special theories (WBQ, WTQ, QHQ) of non-Fregean logic which determine correspondingly special elementary, i.e. axiomatic, strengthenings of that logic. The constructions presented here extend what has been done in [1] and [2] within the SCI-language. They also apply to the non-Fregean logic in comprehensive languages of kind W involving quantifiers binding sentential variables and nominal variables, as well, (compare [3], [4]). However, for the sake of simplicity, the underlying language considered here is the SCI-language with quantifiers $\forall$ and $\exists$. The reader may inspect the case when the numerical quantifiers also occur in the language.

The language L has the following symbols: (a) sentential variables p_k for $k = 1, 2, \ldots$, written sometimes as p, q, r, s; (b) truth-functional connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$, and the identity connective $\equiv$; (c) quantifiers $\forall, \exists$ binding sentential variables; (d) auxiliary symbols $1, 0, \Box, \leqslant$ equationally defined.

Let Cn the non-Fregean consequence operation restricted to formulas of L; for details see [4]. It is generated by the Modus Ponens rule and logical axioms divided into three sets:

$A1$ = axioms for truth-functional connectives;
$A2$ = axioms for quantifiers;
$A3$ = axioms for the identity connective.

One obtain the subconsequence Cn_0 of Cn by ignoring all axioms in $A3$. If A is a set of formulas, then $Cn(A)$ is the set of all generalizations of formulas in A, and $Eqv(A)$ is the set all equations $\alpha \equiv \beta$ such that $(\alpha \Leftrightarrow \beta) \in A$.

*I owe much to conversations with Prof. Suszko.

Let the set D_1 be constituted by the four formulas:

$$\begin{array}{rcl} \forall p(0 \equiv (p \wedge \neg p)), & & \forall p(1 \equiv (p \vee \neg p)) \\ \forall p \forall q((p \leqslant q) & \equiv & ((p \Rightarrow q) \equiv 1)) \\ \forall p(\Box p & \equiv & (p \equiv 1)) \end{array}$$

§2. First, we define the invariant theory WBQ which corresponds to the smallest Boolean theory in SCI, WB.

$$WBQ =_{df} Cn(Gn(Eqv(Cn_0(\emptyset))) \cup D_1)$$

Then, we define two simpler axiom systems of the theory WBQ. Let AB be the set of six formulas:

$$\begin{array}{rcl} \forall p \forall q \forall r(((p \vee q) \wedge r) & \equiv & ((q \wedge r) \vee (p \wedge r))) \\ \forall p \forall q \forall r(((p \wedge q) \vee r) & \equiv & ((q \vee r) \wedge (p \vee r))) \\ \forall p \forall q((p \vee (q \wedge \neg q)) & \equiv & p) \\ \forall p \forall q((p \wedge (q \vee \neg q)) & \equiv & p) \\ \forall p \forall q((p \Rightarrow q) & \equiv & (\neg p \vee q)) \\ \forall p \forall q((p \Leftrightarrow q) & \equiv & ((p \Rightarrow q) \wedge (q \Rightarrow p))) \end{array}$$

This is an axioms system for Boolean algebras written by means of truth-functional connectives and sentential variables (compare [5]). Subsequently, let Q be the set of all generalizations of formulas of the following form:

$$\begin{array}{rcl} \forall p_i \alpha \leqslant \alpha[p_i/\beta], & & \alpha[p_i/\beta] \leqslant \exists p_i \alpha \\ \forall p_j(\forall p_i(p_j \leqslant \gamma) & \Rightarrow & (p_j \leqslant \forall p_i \gamma)) \\ \forall p_j(\forall p_i(\gamma \leqslant p_j) & \Rightarrow & (\exists p_i \gamma \leqslant p_j) \end{array}$$

where α, β, γ are any formulas, $i \neq j$ and p_j is not free in γ. The set Q corresponds to the Q-principle introduced in [3]. Finally, let $A2*$ be the set of all generalizations of formulas of the following form:

$$\begin{array}{ll} \forall p \alpha \leqslant \alpha[p/\beta], & \forall p(\alpha \Rightarrow \beta) \leqslant (\forall p \alpha \Rightarrow \forall p \beta) \\ \alpha[p/\beta] \leqslant \exists p \alpha, & \forall p(\alpha \Rightarrow \beta) \leqslant (\exists p \alpha \Rightarrow \exists p \beta) \\ \alpha \leqslant \forall p \alpha, \exists p \alpha \leqslant \alpha & (p \text{ is not free in } \alpha) \end{array}$$

Notice that the formulas in $A2*$ arise from those in $A2$ by replacing the main connective $\Rightarrow$ by the ordering connective $\leqslant$.

THEOREM 1. $WBQ = Cn(AB \cup Q \cup D_1) = Cn(AB \cup A2 * \cup D_1)$.

§3. The invariant theory WTQ is defined as the set

$$Cn(Cn(Eqv(Cn(\emptyset))) \cup D_1)$$

It is an extension of WBQ, i.e. $WBQ \subset WTQ$. We are going to present two more intuitive axiom systems of the theory WTQ. Let D_2 be the set constituted by the four formulas:

$$1 \equiv \exists p, \qquad 0 \equiv \forall pp, \qquad \forall p(\Box p \equiv (p \equiv 1))$$

$$\forall p \forall q((p \leqslant q) \equiv ((p \Rightarrow q) \equiv 1)$$

Consequently, let $A3*$ be the set consisting of the following formulas:

(1) $\forall p \forall q((p \equiv q) \leqslant (p \Leftrightarrow q)$
(2) $\forall p \forall q((p \equiv q) \leqslant (\neg p \equiv \neg q)$
(3) $\forall p \forall q \forall r \forall s(((p \equiv q) \wedge (r \equiv s)) \leqslant ((p \S r) \equiv (q \S s)))$

where § is $\wedge, \vee, \Rightarrow, \Leftrightarrow, \equiv$, and all generalizations of formulas of the form:

(4) $\forall p_i(\alpha \equiv \beta) \leqslant (K_{p_i}\alpha \equiv K_{p_i}\beta)$
(5) $1 \equiv (\alpha_1 \equiv \alpha_2)$

where K is $\forall$ or $\exists$ and α_1, α_2 differ at most in bound variables. Observe that $A3*$ and $A3$ are related like $A2*$ and $A2$.

THEOREM 2. $WTQ = Cn(AB \cup A2 * \cup A3 * \cup D_2)$.

PROOF. The inclusion $AB \cup A2 * \cup A3* \subset WTQ$ is nearly obvious. For the other part of the proof observe that the theory $Cn(AB \cup A2 * \cup A3 * \cup D_1)$ is invariant and closed under the G-rule $\alpha / \Box\alpha$. Hence, it is closed under the QF-rule $\alpha \Leftrightarrow \beta / \Box(\alpha \equiv \beta)$.

Let IQ be the set constituted by the five formulas:

(6) $\forall p \forall q(\Box(p \Leftrightarrow q) \equiv (p \equiv q))$
(7) $\Box 1 \equiv 1$
(8) $\forall p(\Box p \leqslant p)$
(9) $\forall p \forall q(\Box(p \wedge q) \equiv (\Box p \wedge \Box q))$
(10) $\forall p(\Box\Box p \equiv \Box p)$

and all generalizations of equations of the form:

(11) $\Box \forall p \alpha \equiv \forall p \Box \alpha$

THEOREM 3. $WTQ = Cn(AB \cup Q \cup IQ \cup D_1)$.

PROOF. It is enough to prove two inclusions: $A3* \subset Cn(WBQ \cup IQ)$ and $IQ \subset Cn(WBQ \cup A3*)$. The case of formulas (1), (2), (3) in $A3*$ is easy: compare [2]. For the formulas (4), (5) one may use the following derivations:

(a)
$$\begin{array}{rl} \alpha_1 \equiv \alpha_2 & \text{by } A3 \\ \Box(\alpha_1 \Leftrightarrow \alpha_2) & \text{by (6)} \\ \Box\Box(\alpha_1 \Leftrightarrow \alpha_2) & \text{by (10)} \\ \Box(\alpha_1 \equiv \alpha_2) & \text{by (6)} \end{array}$$

(b)
$$\begin{array}{rcll} \forall p(\alpha \Leftrightarrow \beta) & \leqslant & (K_p\alpha \Leftrightarrow K_p\beta) & \text{by } A2* \text{ and } Q \\ \Box\forall p(\alpha \Leftrightarrow \beta) & \leqslant & \Box(K_p\alpha \Leftrightarrow K_p\beta) & \text{by (9)} \\ \forall p(\alpha \equiv \beta) & \leqslant & (K_p\alpha \equiv K_p\beta) & \text{by (6), (11)} \end{array}$$

For the second inclusion, observe first that the formulas (6), (7), (8), (9), (10) in IQ may easily be deduced from (1), (2), (3) and (5) in $A3*$ by reasonings used in [6]. By (4) in $A3*$ we directly have:
To get formula (11) in IQ we use the derivation:

$$\begin{array}{rl} \forall q(\forall p\alpha \leqslant \alpha[p/q]) & \text{by } Q \\ \forall q(\Box\forall p\alpha \leqslant \Box\alpha[p/q]) & \text{by (9) in } IQ, \text{ already obtained} \\ \Box\forall p\alpha \leqslant \forall q\Box\alpha[p/q]) & \text{by } Q \\ \Box\forall p\alpha \leqslant \forall p\Box\alpha & \text{by } A3 \end{array}$$

REMARK. The last formula in D_1 follows immediately (by AB) from the formula (6) in IQ.

§4. Consider the following formulas:

(12) $\forall p\forall q(((p \equiv q) \equiv 1) \vee ((p \equiv q) \equiv 0))$
(13) $\forall p\forall q((p \equiv q) \equiv ((p \equiv q) \equiv 1))$
(14) $\forall p\forall q(\neg(p \equiv q) \equiv ((p \equiv q) \equiv 0))$
(15) $\forall p((\Box p \equiv 1) \vee (\Box p \equiv 0))$

Let AH be the set AB supplemented with the formula (12). Similarly, let AH_1 and AH_2 be the set AB together with two formulas (13), (14) or (6), (15), correspondingly. It is easy to see that:

$$Cn(AH \cup D_1) = Cn(AH_1 \cup D_1) = Cn(AH_2 \cup D_1)$$

(compare [2]).

The invariant theory WHQ is defined as the set $Cn(AH \cup Q \cup D_1)$.

It follows that WHQ is an extension of WTQ, that is, $WTQ \subset WHQ$.

One may be eager to compare our theories, defined above, with the quantified modal systems considered in [7].
The theories WTQ and WHQ may be identified with $S4\Pi$ and $S5\Pi$, respectively. On the other hand it seems, that no quantified modal system, known in the literature, corresponds to the theory WBQ.

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