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WEAKEST NORMAL CALCULI WITH RESPECT TO M^n-COUNTERPARTS

This is an abstract of the paper submitted to Studia Logica.

By M^n-counterpart of any modal system we mean the set of all formulas which, while preceded n-times by sign M, become theses of the system. In [1], for some normal modal systems the greatest normal modal systems with equal M^n-counterparts were constructed and axiomatized. In this paper, for some normal modal systems we axiomatized the weakest normal modal systems with equal M^n-counterparts.

We use the well-known logical and set-theoretical notation. The symbol ω denotes the set of natural numbers; the elements of this set will be denoted by k, m, n. The logical connectives will be represented by $\rightarrow$, L, M, denoting material implication, necessity, and possibility, respectively. Propositional variables will be represented by p, q, … and formulas by capitals A, B, …. By FOR we denote the set of all formulas. We put

$$L^0A = A,\ L^{n+1}A = LL^nA,\ M^0A = A,\ M^{n+1}A = MM^nA.$$

Let PC denote the set of all classical tautologies. Cn_R is a consequence operator defined by PC and a set of rules of deduction, whereas Cn_{R_0} is defined by means of PC, substitution, detachment and Gödel's rule: if A, then LA. Logical systems are treated as sets of formulas.
Let

$K = Cn_{R_0}(L(p \rightarrow q) \rightarrow (Lp \rightarrow Lq))$,
$D = Cn_{R_0}(K, M(p \rightarrow p))$.

As is well known, the system K (see [2]) is the smallest normal modal system and D is a deontic system of Lemmon (see [4]). By $\mathbb{K}(\mathbb{D})$ we denote the class of all normal modal systems including K (including D). Let $X \subset FOR$ and $n \in \omega$. We put

$M^n(X) = \{\alpha \in FOR : M^n\alpha \in X\}$,
$(X)M^n = \{M^n\alpha \in X : \alpha \in FOR\}$.

THEOREM 1. *For every $X \in \mathbb{K}$ and $n \in \omega$ the following conditions are equivalent:*

(1) $X \subset M^n(X)$,
(2) $D \subset X$,
(3) $M^n(X) \neq \emptyset$.

The proof is analogous to that of Theorem 4 in [5].

COROLLARY 2. *D is the weakest normal modal system for which $M^n(D) \neq \emptyset, n \in \omega$.*

Notice that considering M^n-counterparts of normal modal systems it is enough to confine the considerations to the systems belonging to the class $\mathbb{D}$.

THEOREM 3. *Let $X \in \mathbb{D}$. Then $Cn_{R_0}((X)M^n)$ is the smallest normal modal system such that*

$$M^n(Cn_{R_0}((X)M^n)) = M^n(X).$$

As I was informed the theorem has been proved by J. Perzanowski but up to now it has not been published.

Notice that Theorem 3 yields that for each normal modal system X belonging to $\mathbb{D}$ there exists the smallest modal system with M^n-counterpart the same as that of X. The system will be denoted by X_{M^n}. Set $(X)M^n$ is always infinite, thus the axiomatics of X_{M^n} given in Theorem 3 is an infinite one. J. Kotas and N. C. A. da Costa in [3] formulated the problem of finite axiomatization of system X_{M^n} with the assumption of finite axiomatizability of X.

We shall use the following deduction rules:

(r_1^{nk}) : If M^nL^kA, then $M^nL^{k+1}A$,
(r_2^{nk}) : If M^nL^kA, $M^nL^k(A \to B)$, then M^nL^kB,
(r_3^{nk}) : If $M^nL^kM^nA$, then M^nA.

DEFINITION 4 ([1]). Let $k, n \in \omega$,

(1) $\mathcal{D}_n^k = \{X \in \mathbb{D} : X$ is closed under the rules (r_i^{nk}), $i = 1, 2, 3\}$,
(2) $\mathcal{D}_n = \bigcup_{k \geqslant 1} \mathcal{D}_n^k$.

Observe that if $X \in \mathcal{D}_n$, then there exists a natural number k such that $X \in \mathcal{D}_n^k$. Let $k(X)$ denote one of those natural numbers for which $X \in \mathcal{D}_n^{k(X)}$.

Let us confine our considerations to the family of normal modal systems X such that $X \in \mathcal{D}_n$.

Let X be any normal modal system. It is known that for X there exists a set A_X of axioms (finite or infinite) and a finite set R_X of rules of deduction such that $X = Cn_{R_X}(A_X)$. Notice that without any loss of generality of considerations we may assume that R_X contains merely the detachment rule for material implication, substitution and Gödel's rule. Thus we can assume that $X = Cn_{R_0}(A_X)$.

Theorem 5. *Let $X \in \mathcal{D}_n$. Then*
$X_M n = Cn_R(M^n L^{k(X)} A_X)$, *where*
$M^n L^{k(X)} A_X = \{M^n L^{k(X)} \alpha : \ \alpha \in A_X\}$ *and*
$R = R_0 \cup \{r_1^{nk(X)}, r_2^{nk(X)}, r_3^{nk(X)}\}$.

From the theorem we immediately have

Corollary 6. *Let $X \in \mathcal{D}_n$. If $X = Cn_{R_0}(A_X)$ is finitely axiomatizable, then X_{M^n} is also finitely axiomatizable.*

Theorem 5 and Corollary 6 constitute a partial solution of the problem formulated in [3].

References

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