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THE EXISTENCE OF MATRICES STRONGLY ADEQUATE FOR E, R AND THEIR FRAGMENTS

This is an abstract of a lecture read at the seminar of the Logic Department, Wrocław University, April 1978.

I. Let $\underline{L} = (L, \rightarrow, \vee, \wedge, \sim)$ by the usual sentential language with an infinite set $v(L)$ of sentential variables; let $\underline{L}^+ = (L^+, \rightarrow, \vee, \wedge)$ and $\underline{L}^I = (L^I, \rightarrow)$ be the positive and pure implicational fragments of $\underline{L}$ respectively; let $\underline{S} = (S, \ldots)$ be a variable ranging over $\underline{L}, \underline{L}^+, \underline{L}^I$. For any $X \subseteq S$, $v(X)$ denotes the set of all variables occurring in the formulas in S. Endomorphisms of $\underline{S}$ are called *substitutions*; for $X \subseteq S$ put $SbX = \{e\alpha : \alpha \in X$ and $e : \underline{S} \rightarrow^{hom} \underline{S}\}$. E and R are the sets of theorems of the well-known ANDERSON and BELNAP's systems: of *entailment* (cf. [1]) and of *relevant implication* (cf. [2]) respectively; E^+, R^+, E^I, R^I are their positive and pure implicational fragments. $\mathbf{2}$ is the set of all 2-valued tautologies, $\mathbf{2}^I$ is the set of all pure implicatonal 2-valued tautologies. For any $X \subseteq L$, define $C_E(X)$ $(C_R(X))$ to be the smallest set of formulas including $X \cup E$ $(X \cup R)$ and closed under *modus ponens* and *adjunction*; C_E^+, C_R^+, C_E^I, C_R^I are defined analogously, i.e., for any set X included in the suitable S and for $C \in \{C_E^+, C_R^+, C_E^I, C_R^I\}$ put $C(X)$ to be the smallest set of formulas of S including $X \cup E^+$ (or $X \cup R^+$ or $X \cup E^I$ or $X \cup R^I$) and closed under modus ponens (and adjunction in the case $C = C_E^+$ or $C = C_R^+$). $C_{\mathbf{2}}$, $C_{\mathbf{2}}^I$ are defined in an obvious way. Every operation $C_{\S}^0$ defined is a structural and finitary consequence in $\underline{S}$ (cf. [5]).

By a *matrix* for the language $\underline{S}$ we shall mean any pair $\mathcal{M} = (\underline{M}, B)$, where $\underline{M} = (M, \rightarrow, \ldots)$ is an algebra similar to $\underline{S}$ and $B \subseteq M$ (B is the set of *designated elements* of $\mathcal{M}$). Homomorphism of $\underline{S}$ into $\underline{M}$ are called *valuations*. For any matrix $\mathcal{M}$ let $C_{\mathcal{M}}$ denote the *matrix consequence* (cf. [5]) generated by $\mathcal{M}$ in $\underline{S}$. Where C^1, C^2 are consequences in $\underline{S}$ we set $C^1 \leqslant C^2$ to mean "for all $X \subseteq S$, $C^1(X) \subseteq C^2(X)$". If C is a consequence in $\underline{S}$, then $Matr(C)$ denotes the class of all matrices $\mathcal{M}$ for $\underline{S}$ such that $C \leqslant C_{\mathcal{M}}$.

THEOREM 1. (WÓJCICKI [9]) *If C is structural, then $\alpha \in C(X)$ iff for all $\mathcal{M} \in Matr(C)$, $\alpha \in C_{\mathcal{M}}(X)$.*

$\mathcal{M}$ is said to be a *strongly adequate matrix* for C, if $C = C_{\mathcal{M}}$.

THEOREM 2. (ŁOŚ-SUSZKO [5], WÓJCICKI [8]) *Let C be a structural consequence in $\underline{S}$. Then there exists a matrix strongly adequate for C iff for all $\alpha \in S$, $\underline{H} \subseteq 2^S$, if $X_0 \in \underline{H}$ and*

(i) $C(X) \neq S$, *all* $X \in \underline{H}$,
(ii) $v(X) \cap v(Y) = \emptyset$, *all* $X, Y \in \underline{H}, X \neq Y$,
(iii) $v(\alpha) \cap v(Y) = \emptyset$, *all* $X \in \underline{H} - \{X_0\}$,
(iv) $\alpha \in C(\bigcup \underline{H})$,
then
(v) $\alpha \in C(X_0)$.

II. In this paper we discuss the problem of the existence of strongly adequate matrices for the relevant logics under consideration. We start with investigating $\underline{L}$ and $\underline{L}^+$.

THEOREM 3. (MAKSIMOVA [6]) *Let C be one of the logics C_E, C_R, C_E^+, C_R^+. If $\alpha \in C(X \cup \{\beta\})$ and $v(\beta) \cap v(X \cup \{\alpha\}) = \emptyset$, then $\alpha \in C(X)$.*

It is well-known that for any $C \in \{C_E, C_R, C_E^+, C_R^+\}$ and for any finite X, $C(X) \neq S$. What is more, for any finite X, $C(X) = C(\bigwedge X)$, where $\bigwedge X$ is the conjunction of all formulas in X. Since C is finitary, the following theorem immediately results from Th. 2, via Maksimova's Theorem:

THEOREM 4. *There exists a matrix strongly adequate for C_E (C_R, C_E^+, C_R^+).*

III. In this section we examine pure implicational relevant logics. A set $X \subseteq L^I$ is said to be a *Post-complete extension* of E^I provided that $X = SbX$, $C_E^I(X) \neq L^I$ and for any $\alpha \notin X$, $C_E^I(Sb\alpha) = L^I$.

LEMMA 1. $\mathbf{2}^I$ *is the only Post-complete extension of* E^I.

PROOF. Put $\eta =_{df} ((p \to (p \to p)) \to p) \to [((p \to (p \to p)) \to p) \to \to ((p \to (p \to p)) \to p)]$, $\xi =_{df} (p \to (p \to p)) \to [(p \to (p \to p)) \to \to (p \to (p \to p))]$. Both η and ξ are theorems of E. Let Y be a Post-complete set, with $Y \neq \mathbf{2}^I$. Define a new consequence C^* in $\underline{L}^I$ so that for any $X \subseteq L^I$

$$C^*(X) =_{df} C^I_E(Y \cup X).$$

Clearly $C^*(\mathbf{2}^I) = L^I$; hence $p \in C^*(\mathbf{2}^I)$. Put $eq =_{df} (p \to (p \to p)) \to p$ for every $q \in v(L^I)$. Since $\eta \in E$, then $e\mathbf{2}^I \subseteq E$, and we obtain by structurality of C^*, $ep = (p \to (p \to p)) \to p \in C^*(e\mathbf{2}^I) \subseteq C^*(E) = C^*(\emptyset) = Y$. Put $e_0 p =_{df} p \to (p \to p)$. Thus $e_0((p \to (p \to p)) \to p) = \xi \to (p \to (p \to p)) \in Y$. Finally $p \to (p \to p) \in Y$ and $p \in Y$ which is impossible. The proof is concluded.

LEMMA 2. *If $C^I_E \leqslant C$ for some consistent structural C, then $C \leqslant C^I_{\mathbf{2}}$.*

PROOF. Suppose that $\alpha \in C(X)$ and $\alpha \notin C^I(X)$ for some α, X. There is a substitution e_0 such that $e_0 X \subseteq \mathbf{2}^I$ and $e_0\alpha \notin \mathbf{2}^I$. Suppose that $C(\mathbf{2}^I) = L^I$. Making use of the formulas η, ξ above, one can show that $p \in C(\emptyset)$, which is impossible; thus $C(\mathbf{2}^I) \neq L^I$, and by Lemma 1, $C(\mathbf{2}^I) = \mathbf{2}^I$. By structurality of C we get $e_0\alpha \in C(e_0 X) \subseteq C(\mathbf{2}^I) = \mathbf{2}^I$ – a contradiction.

LEMMA 3. *For any matrix $\mathcal{M} = (\underline{M}, B) \in Matr(C^I_E)$ and for any finite set $X \subseteq L^I$ there is a valuation $h : \underline{L}^I \to^{hom} \underline{M}$ such that $hX \subseteq B$.*

PROOF. Suppose that $hX \not\subseteq B$ for any h. Then $B \neq M$, $C_{\mathcal{M}}(\emptyset) \neq L^I$ and $C_{\mathcal{M}}(X) = L^I$. By Lemma 2, $C_{\mathcal{M}} \leqslant C^I_{\mathbf{2}}$. Thus $C^I_{\mathbf{2}}(X) = L^I$ which is impossible, because every finite set of pure implicational formulas is classically consistent.

THEOREM 5. *There exists a matrix strongly adequate for C^I_E.*

PROOF. Suppose that conditions (i)-(iv) of Theorem 2 are satisfied but $\alpha \notin C^I_E(X_0)$. C^I_E is finitary; thus we can assume that all X's in $\underline{H}$ are finite. By Theorem 1 there is a matrix $\mathcal{M}_0 = (\underline{M}_0, B_0) \in Matr(C^I_E)$ such that for some valuation $h_0 : \underline{L}^I \to^{hom} \underline{M}_0$

$$(*) \qquad h_0 X_0 \subseteq B_0 \text{ and } h_0\alpha \notin B_0.$$

For any $X \in \underline{H} - \{X_0\}$ set h_X to be a valuation of $\underline{L}^I$ in $\mathcal{M}_0$ such that $h_X X \subseteq B_0$; is finite, thus h_X does exist by Lemma 3. Define a new valuation h^* in $\mathcal{M}_0$ as follows: for every $p \in v(L)$.

$$h^*p = \begin{cases} h_0 p & \text{if } p \in v(X_0 \cup \{\alpha\}), \\ h_X p & \text{if } p \in v(X), X \in \underline{H} - \{H_0\}, \\ p & \text{otherwise.} \end{cases}$$

By (ii), (iii) in Theorem 2 we get $h^* \bigcup \underline{H} \subseteq B_0$ and by (∗), $h^*\alpha = h_0\alpha \notin B_0$. Thus $\alpha \notin C_E^I(\bigcup \underline{H})$ contradicting (iv) and concluding the proof.

It immediately follows from Lemma 1 that $\mathbf{2}^I$ is the only Post-complete extension of R^I (cf. also [3]). Thus by replacing E by R in the proof above we obtain

THEOREM 6. *There exists a matrix strongly adequate for* C_R^I.

IV. As to R, the existence of adequate matrices depends in some way on the formalisation. Some authors (cf. e.g. [7]) consider a variant of R, say R^*, defined in the language $\underline{L}^* = (L^*, \to, \vee, \wedge, \sim, f)$, f being a "false" constant; R^* is a conservative extension of R. Analogously to C_E and C_R define C_R^* as follows: for any $X \subseteq L^*$, $C_R^*(X)$ is the smallest set of $\underline{L}^*$-formulas including $X \cup R^*$ and closed under modus ponens and adjunction. Consider a matrix $\mathcal{M}_3^* = ((\{1, 0, -1\}, \to, \vee, \wedge, \sim, \overline{f}), \{0, 1\})$ where $x \vee y = max(x, y); x \wedge y = min(x, y); \sim x = -x$; $\overline{f} \equiv 0$; $x \to y = -x \vee y$ if $x \leqslant y$ and otherwise $x \to y = -x \wedge y$. ($\mathcal{M}_3$ is the 3-element SUGIHARA matrix (cf. [4]) with a new nullary operation $\overline{f}$.) all the axioms and rules of C_R^* are satisfied in $\mathcal{M}_3^*$. Thus $C_R^* \leqslant C_{\mathcal{M}_3^*}$ and we obtain

$$(**) \qquad C_R^*(f) \neq L^*.$$

Now suppose that there is a matrix strongly adequate for C_R^*, i.e. $C_R^* = C_{\mathcal{M}}$ for some $\mathcal{M} = (\underline{M}, B)$, $\underline{M} = (M, \to, \vee, \wedge, \sim, \underline{f})$. Since f is not a theorem of R, we get $\underline{f} \notin B$. Thus $C_R^*(f) = C_{\mathcal{M}}(f) = L^*$, contradicting (∗∗) and concluding the proof of the following

THEOREM 7. *There is no matrix strongly adequate for* C_R^*.

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