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DECIDABILITY OF STRUCTURAL COMPLETENESS FOR STRONGLY FINITE PROPOSITIONAL CALCULI

Let $LNG = (FOR, CON)$ be an absolutely free algebra generated by an infinite set of generators $\{p_1, p_2, \ldots\}$, where CON is a finite sequence of operations denoted by sentential connectives. $LNG^k = (FOR^k, CON)$ is the subalgebra of algebra LNG generated by $p_1, \ldots, p_k$. By M a generalized matrix (cf. Wójcicki [3]) associated with LNG is denoted. The symbol Cn_M denotes a matrix consequence determined by M. A consequence C is strongly finite iff there is a finite matrix M such that $C = Cn_M$. $a \sim_M b$ iff for every homomorphism h from LNG into the algebra of M $ha = hb$ (for all $a, b \in FOR$). Instead of $LNG^k/\sim_M$, $|a| \sim_M$, we write LNG^k/M, a_M, respectively. If M is a k-valued matrix, then the symbol M^+ denotes the matrix $(LNG^k/M, \{a_M : a \in FOR^k \cap Cn_M\emptyset\})$. A calculus (LNG, C) is structurally complete (cf. Prucnal [2]) iff for all $a \in FOR$, $X \subseteq FOR$ the condition: for every substitution $s : LNG \to LNG$ if $sX \subseteq C\emptyset$ then $sa \in C\emptyset$ implies $a \in CX$.

It is obvious that

I. If $a_M = b_M$, then $(sa)_M = (sb)_M$ for every matrix M, every substitution $s : LNG \to LNG$ and all $a, b \in FOR$.

By the proof of Theorem 1 in [3] one can obtain

II. For every k-valued matrix M, all $a \in FOR$, $X \subseteq FOR$, there is a substitution $s : LNG \to LNG^k$ such that

$$\text{if } a \notin Cn_M X, \text{ then } sa \notin Cn_M sX$$

By a generalization of Theorem 19 in [1] or by proof of Theorem 9 in [3] one can obtain

III. If M is a finite matrix, then the calculus (LNG, Cn_{M^+}) is structurally complete and $Cn_M\emptyset = Cn_{M^+}\emptyset$.

Let M be a k-valued matrix. It is obvious that the set FOR^k/M is finite. By [1] it is known that a set of representatives of FOR^k/M can be effectively constructed. This set is denoted by REP^k. If $a \in FOR^k$, then $\overline{a}$ denotes the formula $b \in REP^k$ such that $a_M = b_M$. If $X \subseteq FOR^k$, then $\overline{X} = \{\overline{a} : a \in X\}$.

Theorem. *The structural completeness of a strongly finite propositional calculus is decidable.*

Let M be a k-valued matrix. Since REP^k, M^+ are finite and effectively constructed, there is an effective method which enables us to decide whether the following holds:

(+) for every $X \subseteq REP^k \quad REP^k \cap Cn_{M^+}X \subseteq Cn_M X$

It is obvious that if (LNG, Cn_M) is structurally complete, then (+) holds. Then it is sufficient for the proof of the theorem to show that (+) implies the structural completeness of (LNG, Cn_M). Suppose that (+) holds and the are $a \in FOR$, $X \subseteq FOR$ such that

1. for every substitution $s : LNG \to LNG$

$$\text{if } sX \subseteq Cn_M\emptyset, \text{ then } sa \in Cn_M\emptyset$$

and

$$a \notin Cn_M X$$

By II, there is a substitution $s : LNG \to LNG^k$ such that $sa \notin Cn_M sX$. Hence $\overline{sa} \notin Cn_M \overline{sX}$. By (+), $\overline{sa} \notin Cn_{M^+}\overline{sX}$. By III and the definition of structural completeness there is a substitution $s_1 : LNG \to LNG$ such that $s_1\overline{sX} \subseteq Cn_{M^+}\emptyset$ and $s_1\overline{sa} \notin Cn_{M^+}\emptyset$. By III, $s_1 s \subseteq Cn_M\emptyset$ and $s_1 sa \notin Cn_M\emptyset$. This contradicts 1.

The symbol C_R denotes the consequence obtained by adding to the rules of C the set of rules R.

Corollary. *If a calculus (LNG, C) is strongly finite, then there is a finite set of standard rules R such that the calculus (LNG, C_R) is structurally complete and $C\emptyset = C_R\emptyset$.*

Let M be a k-valued matrix such that $C = Cn_M$ and let $SEQ(REP^k)$ be the set of all sequents $\langle a_1, \ldots, a_i, a \rangle$ such that $a, a_1, \ldots, a_i \in REP^k$, $a \in Cn_{M^+}\{a_1, \ldots, a_i\}$ and $a \notin Cn_M\{a_1, \ldots, a_i\}$. Let $STAND(SEQ(REP^k))$ be the set of all standard rules (cf. [3]) determined by the sequents of $SEQ(REP^k)$. Then it is obvious that for $R = STAND(SEQ(REP^k))$ (+) holds. This complete the proof of the corollary.

References

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[2] T. Prucnal, *On structural completeness of some pure implicational calculi*, **Studia Logica** 30 (1972), pp. 45–50.

[3] R. Wójcicki, *Strongly finite sentential calculi*, [in:] **Selected papers on Łukasiewicz sentential calculi**, Wrocław, 1977, pp. 53–77.

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