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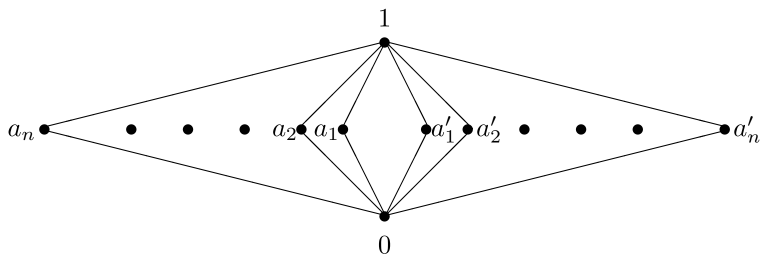
ON SOME MATRIX OF THE BIRKHOFF AND V. NEUMANN QUANTUM LOGIC

In [4] Piron investigates lattices of sentences of some physical experiments. In the case of experiments of classical physics the lattices are Boolean algebras. Particular attention should be given to results obtained in the case of some typical quantum experiments. For various experiments we obtain simple modular and orthocomplementary lattices. Every such lattice can be described by the following diagram:



where $n = 1, 2, \dots$.
Let us consider a class of matrices

$$\mathcal{M}_{2n,i} = \langle A_{2n}, \{1\}, \cup, \cap, ', \rightarrow_i \rangle,$$

where $A_{2n} = \{1, 0, a_1, a'_1, \dots, a_n, a'_n\}$ and operations $\cup, \cap, '$ are the same as lattice operations given by the diagram above. Operations $\rightarrow_i$ ($i = 0, 1, \dots, 4$) are defined by the following formulas:

$a \rightarrow_0 b = (a \cap b) \cup (a' \cap b) \cup (a' \cap b')$,
$a \rightarrow_1 b = (a' \cup b) \cap (a \cup (a' \cap b) \cup (a' \cap b'))$,
$a \rightarrow_2 b = (a' \cap b') \cup b$,

$a \to_3 b = a' \cup (a \cap b)$,
$a \to_4 b = (a' \cup b) \cap (b' \cup (a \cap b) \cup (a' \cap b))$.

With the symbol $\mathcal{M}_{\omega.i}$ we denote a matrix in which the set A_ω is infinite and the operations $\cup, \cap, '$ are defined as in the matrices $\mathcal{M}_{2n.i}$. Sets of tautologies of each of these matrices define some logical systems. Every such system is an intermediate logic between classical logic and Birkhoff and v. Neumann quantum logic.

As it can be seen from introductory remarks, these logics may be helpful in analysing results of some physical experiments. It should be noticed that to every experiment of quantum mechanics one can attribute a set of basic concepts and a set of sentences about these basic concepts. It is the theory of the experiment. One can assume that the metatheory of the mentioned experiments is based on the logics designed by matrices $\mathcal{M}_{2n.i}$ and $\mathcal{M}_{\omega.i}$.

It seems that the problem of axiomatization of these logics is a matter of relative importance. For this purpose let us consider the following set of formulas:

$A1.$ $P \to (Q \vee \neg Q)$,

$A2.$ $P \to \neg\neg P$,

$A3.$ $P \to P \vee Q$,

$A4.$ $P \vee Q \to Q \vee P$,

$A5.$ $P \vee (Q \vee R) \to (P \vee Q) \vee R$,

$Df1.$ $P \wedge Q = \neg(\neg P \vee \neg Q)$,

$A6.$ $P \vee (P \wedge Q) \to P$,

$A7.$ $(P \vee Q) \wedge (P \vee R) \to P \vee (Q \wedge (P \vee R))$,

$A8.0$ $(P \to Q) \to (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$,

$A8.1$ $(P \to Q) \to (\neg P \vee Q) \wedge (P \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q))$,

$A8.2$ $(P \to Q) \to \neg(P \vee Q) \vee Q$,

$A8.3$ $(P \to Q) \to \neg P \vee (P \wedge Q)$,

$A8.4$ $(P \to Q) \to (\neg P \vee Q) \wedge (\neg Q \vee (P \wedge Q) \vee (\neg P \wedge Q))$,

$A9.0$ $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \to (P \to Q)$,

$A9.1$ $(\neg P \vee Q) \wedge (P \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \to (P \to Q)$, *A*9.2 $\neg(P \vee Q) \vee Q \to (P \to Q)$,

*A*9.3 $\neg P \vee (P \wedge Q) \to (P \to Q)$,

*A*9.4 $(\neg P \vee Q) \wedge (\neg Q \vee (P \wedge Q) \vee (\neg P \wedge Q)) \to (P \to Q)$,

Θ $P \wedge (Q \vee (R \wedge S)) \wedge (R \vee S) \to Q \vee (P \wedge R) \vee (P \wedge S)$,

Ξ_n $P \wedge \bigwedge_{1 \leqslant l < j \leqslant 2n} (Q_l \vee Q_j) \to \bigvee_{1 \leqslant l \leqslant 2n} (P \wedge Q_l)$,

Let us denote by $\mathbb{R}$ the set of the following rules:

r.1 $\dfrac{P, P \to Q}{Q}$,

r.2 $\dfrac{P \to Q}{\neg Q \to \neg P}$,

r.3 $\dfrac{P \to Q}{R \vee P \to R \vee Q}$,

r.4 $\dfrac{P \to Q, Q \to R}{P \to Q}$,

The set $\{A1, A2, \ldots, A7, A8.i, A9.i\}$ will be denoted by Ax_i, where $i = 0, 1, 2, 3, 4$.

THEOREM 1. *The pair* $\langle \mathbb{R}, Ax_i \rangle$ *is the axiomatization of Birkhoff and v. Neumann quantum logic extended by the definition of the functor* $\to_i$*,* $i = 0, 1, 2, 3, 4$.

THEOREM 2. *For every* $n \in N$*,* $i = 0, \ldots, 4$*:*

(1) $Cn(\mathbb{R}, Ax_i \cup \{\Theta\}) = E(\mathcal{M}_{\omega.i})$,

(2) $Cn(\mathbb{R}, Ax_i \cup \{\Theta, \Xi\}) = E(\mathcal{M}_{2n.i})$.

References

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