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ON FRIEDMAN'S PROBLEM IN MATHEMATICAL LOGIC*
(Preliminary Report)

0. Let $\mathcal{F}_\Box = \langle F_\Box, \wedge, \sim, \Box\rangle$ be the free algebra in the class of all algebras of the type $\langle 2, 1, 1\rangle$ free-generated by the set $V = \{p_1, p_2, \ldots\} = \{p_i : i \in N\}$. By h^e we denote the extension of the function $e : V \to F_\Box$ to the endomorphism of the algebra $\mathcal{F}_\Box$.

H. Friedman in [1] conjectured that there are sets $M \subseteq F_\Box$ such that:

$(F1)$ $V \subseteq M$,
$(F2)$ $\sim\alpha \in M \Leftrightarrow \alpha \notin M$,
$(F3)$ $\alpha \wedge \beta \in M \Leftrightarrow \alpha \in M \mathbin{\&} \beta \in M$,
$(F4)$ $\Box\alpha \in M \Leftrightarrow \bigvee_{e:V\to F_\Box} h^e(\alpha) \in M$,

for every $\alpha, \beta \in F_\Box$.

In this paper we will show that there exists a set $M \subseteq F_\Box$ such that the conditions $(F1) - (F4)$ are satisfied.

We shall use the symbols: $\Leftrightarrow, \Rightarrow, \mathbin{\&}, \mathbin{\vee\!\!\vee}$ as the well-known propositional connectives from metalanguage. The symbols $\forall$ and $\exists$ will also be used as quantifiers from metalanguage.

1. Let now $\mathcal{F} = \langle F, \vee, \wedge, \to, \sim\rangle$ be the free algebra in the class of all algebras of the type $\langle 2, 2, 2, 1\rangle$ free-generated by the set V. By T we denote the well-known McKinsey-Tarski transformation (cf. [2]), which maps F into $F_\Box$ in the following way:

$a.$ $T(p_i) = \Box p_i$,
$b.$ $T(\sim\alpha) = \Box \sim T(\alpha)$,

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c. $T(\alpha \wedge \beta) = T(\alpha) \wedge T(\beta)$,
d. $T(\alpha \vee \beta) = \sim [\sim T(\alpha) \wedge \sim T(\beta)]$,
e. $T(\alpha \to \beta) = \Box \sim [T(\alpha) \wedge \sim T(\beta)]$
for every $i \in N$ and $\alpha, \beta \in F$.

By INT we mean the set of all theorems of intuitionistic propositional logic, and by $S4$ – the set of all theorems of modal logic. $Cn_{INT}(X)$ is the smallest set containing $INT \cup X \subseteq F$ and closed under the modus ponens rule. Similarly: $Cn_{S4}(Y)$ is the least set containing $Y \cup S4 \subseteq F_{\Box}$ and closed under the modus ponens rule: $\sim (\alpha \wedge \sim \beta)$, α/β.

We have:

Lemma 1. (Cf. [7]). *For every $\gamma \in F$ and $X \subseteq F$:*

$$\gamma \in Cn_{INT}(X) \Leftrightarrow T(\gamma) \in Cn_{S4}(T(X)),$$

where $T(X)$ is the image of the set X.

By F_T we denote the least set containing the image $T(F)$ and closed with respect to: $\wedge, \sim$, and $\Box$.

Lemma 2. (Cf. [6]). *For every $\alpha \in F_T$ there are $\gamma_1, \gamma_2, \ldots, \gamma_n$, $\delta_1, \delta_2, \ldots, \delta_n \in F$ such that:*

$$\alpha \equiv \sim [T(\gamma_1) \wedge \sim T(\delta_1)] \wedge \ldots \wedge [T(\gamma_n) \wedge \sim T(\delta_n)],$$

where $\alpha \equiv \beta \Leftrightarrow_{df} \sim (\alpha \wedge \sim \beta) \wedge \sim (\beta \wedge \sim \alpha) \in S4$.

Let B be the least set containing $\{\sim p_i : i \in N\}$ and closed under the connectives: $\vee, \wedge, \to$, and $\sim$.

Putting

$$ML =_{df} \{\gamma \in F : \forall_{e:V \to B} h^e(\gamma) \in KP\},$$

where KP is an intermediate logic obtained by adding to INT the axioms: $(\sim \gamma \to \alpha \vee \beta) \to (\sim \gamma \to \alpha) \vee (\sim \gamma \to \beta)$, $\alpha, \beta \in F$, we obtain an intermediate logic such that $KP \not\subseteq ML$.

We have:

Lemma 3. (Cf. [3]). *For every $\alpha, \beta \in F$:*

$$\alpha \vee \beta \in ML \Leftrightarrow \alpha \in ML \veebar \beta \in ML.$$

This ML has also the following property[1]:

LEMMA 4. (Cf. [5]). *For every $\alpha, \beta \in F$:*

$$\alpha \to \beta \in ML \Leftrightarrow \forall_{e:V\to F}[h^e(\alpha) \in ML \Rightarrow h^e(\beta) \in ML].$$

Putting

$$ML^{(T)} =_{df} Cn_{S4}(T(ML)),$$

we obtain:

LEMMA 5.

(i) $\gamma \in ML \Leftrightarrow T(\gamma) \in ML^{(T)}$, *for every* $\gamma \in F$.
(ii) $\alpha \in ML^{(T)} \Leftrightarrow \Box\alpha \in ML^{(T)}$, *for every* $\alpha \in F_\Box$.
(iii) $\sim(\sim\Box\alpha \wedge \sim\Box\beta) \in ML^{(T)} \Leftrightarrow \Box\alpha \in ML^{(T)} \Cup \Box\beta \in ML^{(T)}$, *for every* $\alpha, \beta \in F_T$.

Let now $ML^\Box$ be a set defined as follows:

$$ML^\Box =_{df} \{\alpha \in F_\Box : \forall_{e:V\to F_T} h^e(\alpha) \in ML^{(T)}\}.$$

LEMMA 6. *For every $\gamma \in F$:*

$$\gamma \in ML \Leftrightarrow T(\gamma) \in ML^\Box.$$

LEMMA 7. *For every $\alpha, \beta \in F_\Box$:*

(i) $S4 \subsetneq ML^\Box$,
(ii) $\alpha, \sim(\alpha \wedge \sim\beta) \in ML^\Box \Rightarrow \beta \in ML^\Box$,
(iii) $\alpha \in ML^\Box \Rightarrow \forall_{e:V\to F_\Box} h^e(\alpha) \in ML^\Box$,
(iv) $\alpha \in ML^\Box \Leftrightarrow \Box\alpha \in ML^\Box$,
(v) $\Box\alpha \in ML^\Box \Cup \Box\beta \in ML^\Box \Leftrightarrow \sim(\sim\Box\alpha \wedge \sim\Box\beta) \in ML^\Box$.

We define now a set $A \subseteq F_\Box$ in the following way:

$$\beta \in A \Leftrightarrow \exists_{e:V\to F_\Box}\exists_{\alpha\in F-ML^\Box}\beta = \sim(\Box\alpha \wedge h^e(\alpha)),$$

for every $\beta \in F_\Box$.

Thus:

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[1]Let us note that Lemma 4 states that the calculus ML is structurally complete in the sense of W. A. Pogorzelski [4].

LEMMA 8. $Cn_{S4}(ML^{\Box} \cup A \cup V \cup \{\sim T(\gamma) : \gamma \in F - ML\}) \neq F_{\Box}$.

Let $M_0 =_{df} Cn_{S4}(ML^{\Box} \cup A \cup V \cup \{\sim T(\gamma) : \gamma \in F - ML\})$ and let M_* be the maximal element in $\{M \subseteq F_{\Box} : M_0 \subseteq M = Cn_{S4}(M) \neq F_{\Box}\}$. Thus we have:

THEOREM.[2] *The set M_* satisfies the conditions $(F1) - (F4)$.*

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[2]Dr J. Perzanowski informed me that an analogous results had been obtained by Kit Fine (unpublished).