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ON UNIVERSAL ALGEBRAIC LOGIC AND CYLINDRIC ALGEBRAS

This is an abstract of the dissertation [1] which solved some problems raised in [2]. The subject is General Algebraic Logic in the sense of Rasiowa [5], but now for first order logics. Here we discuss the algebraic problems; their connections with (nonclassical and classical) logics were explained in [2]. The variety of cylindric algebras [4] was introduced for the classical first order logic; the present general algebraic (universal algebraic) approach is a generalization of the theory of that variety [4] to make it applicable to other first o. logics as well (cf. Freeman [6]).

Throughout, α, β, γ denote *infinite* ordinals, ω is the set of natural numbers, and Ord is the class of *in finite* ordinals.

Definition 1.1:

1. By a *type-scheme* we understand a quadruple $t = \langle T, \delta, \tau, c\rangle$ where T is a set, $\delta : T \to \omega$, $\tau : T \to \omega$, $c \in T$ and $\delta(c) = \tau(c) = 1$.
2. A type-scheme t defines a *similarity type* t_α for each infinite ordinal α as follows:
   $t_\alpha : \Omega_\alpha \to \omega$, where the set Ω_α of operation symbols is:
   $\Omega_\alpha =^d \{f_{i_1\ldots i_n} : f \in T,\ i_1, \ldots, i_n \in \alpha,\ n = \delta(f)\}$ with arities: $t_\alpha(f_{i_1\ldots i_n}) =^d \tau(f)$.
   (Here $f_{i_1\ldots i_n}$ stands for the $n+1$-tuple $(f, i_1, \ldots, i_n)$.)

Example ([4]): The similarity type of α-dimensional cylindric algebras is: $t_\alpha = \{(\cdot, 2), (-, 1), (c_i, 1), (d_{ij}, 0) : i, j \in \alpha\}$.
The "cylindric type-scheme" consists of $T = \{\cdot, -, c, d\}$ and $\delta(\cdot) = \delta(-) = 0$, $\delta(c) = 1$, $\delta(d) = 2$; $\tau(c) = 2$ etc.
The universe of an algebra $\underline{A}$ is denoted by A.

Definition 1.3. ([4], D. 2.6.1): Let t be a type-scheme, $\underline{A}$ an algebra of type t_α, and let $\xi : \beta \rightarrowtail \alpha$ be arbitrary. Now, $\underline{Rd}^\xi \underline{A}$ is a new algebra of type t_β obtained from A as follows:
The universe of $\underline{Rd}^\xi \underline{A}$ is A.
The interpretation of the operation symbol $f_{i_1 \ldots i_n} \in \Omega_\alpha$ in the new algebra coincides with the interpretation of $f_{\xi(i_1) \ldots \xi(i_n)}$ in the old one, i.e. $f_{i_1 \ldots i_n}^{Rd^\xi A} =^d f_{\xi(i_1) \ldots \xi(i_n)}^{A}$.
$\underline{Rd}^\xi \underline{A}$ is called a *generalized reduct* of A along ξ.
If K is a class of algebras of similarity type t_α, then $Rd^\xi K = \{\underline{Rd}^\xi \underline{A} : \underline{A} \in K\}$.

An element $b \in A$ of an algebra $\underline{A}$ of type t_α is said to be *sensitive* to the index $i \in \alpha$ if b is not a fixed point of the operation c_i^A (if $c_i^A(b) \neq b$).

Definition 1.5 ([4], D. 1.11.1.): An algebra is *locally finite dimensional* if each of its elements is sensitive to finitely many indices only, i.e. iff $(\forall a \in A)(\{i \in \alpha : c_i^A(a) \neq a\}$ is finite). An algebra is *dimension complemented* if to any finite subset B of its universe there are infinitely many indices to which no element of B is sensitive.

Definition 1.6 ([4], D. 2.6.28): Let $\alpha \leqslant \beta$ (i.e. $t_\alpha \subseteq t_\beta$). Let $\underline{B}$ be an algebra of type t_β, and let $\underline{B}'$ be its reduct of type t_α (i.e. we omit the operations which have indices greater than α). An algebra $\underline{A} \subseteq \underline{B}'$ is said to be a *neat subreduct* of $\underline{B}$ if the elements of $\underline{A}$ are not sensitive in $\underline{B}$ to the indices greater than α, i.e. if
$(\forall a \in A)(\forall i \geqslant \alpha) c_i^B(a) = a$.
If K is a class of algebras of similarity type t_β, then $SNr_\alpha K$ denotes the class of those neat subreducts of elements of K, which are of type t_α.

Definition 3.2: By a *system of varieties* of type-scheme t we mean a sequence $\langle V_\alpha \rangle_{\alpha \in Ord}$, for which the following 1.-3. hold:

1. V_α is a variety of type t_α, for every $\alpha \in Ord$.
2. $Rd^\xi V_\alpha \subseteq V_\gamma$ for every inclusion $\xi : \gamma \rightarrowtail \alpha$.
3. For every pair of ordinals $\gamma \leqslant \alpha$ and algebra $\underline{A}$ of type t_α:
   If every generalized reduct of type t_γ of $\underline{A}$ is in V_γ, then the original algebra $\underline{A}$ is in V_α, too (i.e. $[(\forall \xi : \gamma \rightarrowtail \alpha) \underline{Rd}^\xi \underline{A} \in V_\gamma] \Rightarrow \underline{A} \in V_\alpha$.

NOTATION: From now on $\langle V_\alpha \rangle_{\alpha \in Ord}$ stands for an arbitrary system of varieties belonging to some type-scheme t, and
$Vf_\alpha =^d \{\underline{A} \in V_\alpha : \underline{A}$ is locally finite dimensional$\}$
$Vc_\alpha =^d \{A \in V_\alpha : \underline{A}$ is dimension complemented$\}$
$Vn_{\gamma\alpha} =^d SNr_\gamma V_\alpha$.

THEOREM 3.7: *ω is the least ordinal ρ for which it is true that for every system of varieties and ordinal α, the sequence $\langle Vn_{\alpha\alpha+\rho+\nu} \rangle_{\nu \in \omega \cup Ord}$ is constant, i.e. $Vn_{\alpha\alpha+\rho} = Vn_{\alpha\alpha+\rho+\nu}$ for every ordinal ν.*

NOTATION: $Vn_\alpha =^d Vn_{\alpha\alpha+\omega}$.

NOTATIONS: The letters H, S, P, P^r, Up, Sd denote the operators of taking homomorphic images, subalgebras, direct product, reduced products, ultraproducts and sandwich-subalgebras (see [3]), respectively. That is, if K is a class of algebras, then HK denotes the class of all homomorphic images of elements of K, etc.

REMARK: The operators $SdUp$, SUp, SP^r, HSP are known to coincide with the formation of hulls axiomatizable by $\sqcap_2$-formulas ($\forall\exists$-formulas), by universal formulas, by universal Horn-formulas (quasi-identities), and by identities, respectively.

THEOREM 3.14-3.18: *(For any $\langle V_\beta \rangle_{\beta \in Ord}$ and any α :)*

1. $Vf_\alpha \subseteq Vc_\alpha \subseteq Vn_\alpha = SP^r Vn_\alpha \subseteq V_\alpha$.
2. *If $|\alpha| = \omega$. then*
   $\overline{SdUpVf_\alpha = Sd}UpVc_\alpha$
   $SUpVf_\alpha = SUpVc_\alpha = SP^r Vf_\alpha = SP^r Vc_\alpha$,
   $H\ S\ P\ Vf_\alpha = H\ S\ P\ Vc_\alpha$,
   and only these equalities are valid, i.e. there is a system of varieties $\langle V_\beta \rangle_{\beta \in Ord}$ such that the classes Vf_α, Vc_α, $SPVf_\alpha$, $SPVc_\alpha$, $SdUpVf_\alpha$, $SUpVf_\alpha$, $HSPVf_\alpha$, Vn_α, HVn_α, V_α, are all different from one another (for any countable α).
3. *If $\alpha \geqslant \omega^+$, then no equality is valid except*
   $\overline{SUpVf_\alpha = SP^r}Vf_\alpha$:
   There is a system of varieties $\langle V_\beta \rangle_{\beta \in Ord}$ for which the classes Vf_α, Vc_α, $SPVf_\alpha$, $SPVc_\alpha$, $SdUpVf_\alpha$, $SdUpVc_\alpha$, $SUpVf_\alpha$, $SUpVc_\alpha$, $HSPVf_\alpha$, $HSPVc_\alpha$, Vn_α, HVn_α, V_α
   are all different from one another, i.e., for instance $HSPVf_\alpha \neq$

$HSPVc_\alpha$.
Only $SUpVc_\alpha = Sp^rVc_\alpha$ is not yet known to be valid or not.

DEFINITION 4.1: A system of varieties $\langle V_\alpha\rangle_{\alpha\in Ord}$ satisfies the "generating condition", if in every algebra of V_ω, elements sensitive only to finitely many indices generate no element sensitive to all indices ($i \in \omega$). More precisely: $(\forall \underline{A} \in V_\omega)(\forall m \in \Omega_\omega)$ [if $a_1, \ldots, a_n \in A$ are sensitive only to finitely many indices, then $(\exists i \in \omega)c_i(m(a_1, \ldots, a_n)) = m(a_1, \ldots, a_n)$ in $\underline{A}$].

THEOREM 4.5: *Let the system of varieties $\langle V_\alpha\rangle_{\alpha\in Ord}$ satisfy the generating condition. Now, for every $\alpha \in Ord$:*
$SdUpVf_\alpha = SdUpVc_\alpha$
$SUpVf_\alpha = SUpVc_\alpha = SP^rVf_\alpha = SP^rVc_\alpha = Vn_\alpha$
$H\ S\ P\ Vf_\alpha = H\ S\ P\ Vc_\alpha = H\ Vn_\alpha$,
and only these equalities are valid, i.e., there is a system of varieties $\langle V_\alpha\rangle_{\alpha\in Ord}$ satisfying the generating condition, such that the classes Vf_α, Vc_α, $SPVf_\alpha$, $SPVc_\alpha$, $SdUpVf_\alpha$, $SUpVf_\alpha$, $HSPVf_\alpha$, V_α are all different.

REMARK: The cylindric algebras of [4] form a systems of varieties $\langle CA_\alpha\rangle_{\alpha\in Ord}$ satisfying the generating condition; therefore Th. 4.5. applies. Surprisingly, the inequalities of Th. 4.5. also hold for them with the exception that $HVn_\alpha = Vn_\alpha$ is true for cylindric algebras. The following problem is open also for cylindric algebras.

PROBLEM: 1. Find a system of varieties $\langle V_\alpha\rangle_{\alpha\in Ord}$ and a Σ_2-formula φ (i.e. $\varphi \equiv \exists\overline{x}\forall\overline{y}u(\overline{x}\ \overline{y})$) such that $Vf_\alpha \models \varphi$ and $Vc_\alpha \not\models \varphi$ for some countable α. (By Th. 3.14. Vf_α and Vc_α are equivalent w.r.t. Π_2-formulas.)
2. Find $\langle V_\alpha\rangle_{\alpha\in Ord}$ and a first order φ such that $Vf_\alpha \models \varphi$ and $Vc_\alpha \not\models \varphi$ for some countable α. What is the smallest prenex for φ $(\Sigma_2?, \Pi_3?, \Sigma_3?, \ldots)$.
3. Solve the above problems for varieties satisfying the generating condition (and for arbitrary α).

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