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# SOME NOTES ON CHARACTERISTIC CONSEQUENCE OPERATIONS

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## §1. Introduction

By a *sentential language* we shall understand an arbitrary absolutely free algebra $L$ of a fixed finitary similarity type. A family $\Upsilon$ of subsets of $L$ is said to be a *closure system on* $L$ iff $\Upsilon$ is closed under arbitrary intersections, i.e. $\bigcap F \in \Upsilon$ if $F \subseteq \Upsilon$. Clearly $L \in \Upsilon$. A *consequence operations on* $L$ is a function $C : P(L) \rightarrow P(L)$, from the power set of $L$, $P(L)$, into $P(L)$ such that $c1)$ $X \subseteq C(X) = C(C(X))$, $c2)$ if $X \subseteq Y$, then $C(X) \subseteq C(Y)$, $X, Y \subseteq L$.

It is known [BS] that every closure system $\Upsilon$ on $L$ defines a consequence operation $C_\Upsilon$ on $L$ such that $C_\Upsilon(X) = X$ iff $X \in \Upsilon$. A consequence operation $C$ is:

- *finite* if $C(X) = \bigcup\{C(X_f) : X_f \subseteq X$ and $card(X_f) < \aleph_0\}$ for every $X \subseteq L$,
- *L-compact* (compact) provided that if $C(X) = L$, then $C(X_f) = L$ for some finite $X_f \subseteq X$,
- *structural* if for every endomorphism $e$ of $L$ ($e \in End(L)$) $e(C(X)) \subseteq C(e(X))$,
- *N-structural* if for every $e \in End(L)$, if $C(X) = L$, then $C(e(X)) = L$,
- *standard* if $C$ is both finite and structural.

## §2. Characteristic consequences

Let $L$ be an arbitrary sentential language.

DEFINITION 2.1. A consequence operation $0$ is *characteristic* iff for every $X \subseteq L$ if $0(X) \neq X$, then $0(X) = L$.

EXAMPLE 2.2. For every consequence operation $C$ we shall define the operation $0_c$ on $L$ as follows.

$$o)\ 0_c(X) = \begin{cases} X & \text{if } C(X) \neq L \\ L & \text{if } C(X) = L. \end{cases}$$

For every $C$ the operation $0_c$ defined in $o$) is a characteristic consequence operation on $L$. We shall call it the *characteristic consequence of* $C$.

DEFINITION 2.3. Let $0$ be a characteristic consequence. The set $\mathbb{K}_0$ of all consequences $C$ such that $0_c = 0$ is said to be the *characteristic class of* $0$.

Note that the set of all characteristic consequences operations on $L$ forms a complete sublattice of the lattice of all consequences on $L$, with the lattice order defined in 2.5. But in general it is not true for every characteristic class $\mathbb{K}_0$ (compare Theorem 2.7).

Let us introduce some definitions.

Let $Th(C) = \{X \subseteq L : C(X) = X\}$ and let $S(C)$ denote the set of all closure systems $\Upsilon$ which satisfy the following conditions:

i) for every $X \in \Upsilon$, $X \in Th(C)$,
ii) for every $X \subseteq L$, if $C(X) \neq L$, then there exists $Y \in \Upsilon$ such that $X \subseteq Y$ and $Y \neq L$.

By $S^0(C)$ we denote a subset of $S(C)$ such that for every $\Upsilon \in S^0(C)$ the consequence $X_\Upsilon$ such that $Th(C_\Upsilon) = \Upsilon$ is a structural one.

DEFINITION 2.4. A consequence operation $C$ is said to be *correct (structurally correct)* iff $S(C)(S^0(C))$ is closed under arbitrary intersections.

DEFINITION 2.5. Let $C_1, C_2$ be consequence operations. $C_1 \leqslant C_2$ iff for all $X \subseteq L$, $C_1(X) \subseteq C_2(X)$.

LEMMA 2.6. *If* $\mathbb{K}_0$ *is the characteristic class of* $0$*, then:*

*i) if* $C_1, C_2 \in \mathbb{K}_0$ *and* $C_1 \leqslant C \leqslant C_2$*, then* $C \in \mathbb{K}_0$*, for all* $C_1, C_2, C$*,* *ii)* $C \in \mathbb{K}_0$ *iff* $Th(C) \in S(0)$.

THEOREM 2.7. *The following conditions are equivalent*

*ii)* $\mathbb{K}_0$ *is a complete lattice,*

*ii)* $\mathbb{K}_0$ *is a complete sublattice of the lattice of all consequences on* $L$*,*

*iii)* $0$ *is correct.*

PROOF. If $F$ is a family of consequences from $\mathbb{K}_0$, then we shall symbolically denote the least upper bound: in the lattice of all consequences on $L$ by $supF$ and in $\mathbb{K}_0$ (if it exists) by $sup_0F$. Observe that for every $F \subseteq \mathbb{K}_0$, $\bigcap F \in \mathbb{K}_0$ and that for every $C \in \mathbb{K}_0$, $C \geqslant 0$ (because $\bigcap \mathbb{K}_0 = 0$).
i $\to$ ii For every family $F \subseteq \mathbb{K}_0$, $0 \leqslant supF \leqslant sup_0F$. Thus by Lemma 2.6 1) $\mathbb{K}_0$ is a sublattice of the lattice of all consequences on $L$.
ii $\to$ i is trivial.
i $\to$ iii Let $\mathcal{F} \subseteq S(0)$ and let for every $\Upsilon \in \mathcal{F}$, $C_\Upsilon$ be a consequence such that $Th(C_\Upsilon) = \Upsilon$. By Lemma 2.6 ii), $F = \{C_\Upsilon : \Upsilon \in \mathcal{F}\} \in \mathbb{K}_0$, and $sup_0F \in \mathbb{K}_0$. But $Th(sup_0F) = \bigcap_{\Upsilon \in \mathcal{F}} Th(C_\Upsilon) = \bigcap \mathcal{F}$. Using Lemma 2.6 ii) again we obtain $\bigcap \mathcal{F} \in S(0)$.
iii $\to$ i If $F$ is a subset of $\mathbb{K}_0$, then, by Lemma 2.6 ii), $\bigcap_{c \in F} Th(C) \in S(0)$, so $sup_0F \in \mathbb{K}_0$ because $Th(sup_0F) = \bigcap_{c \in F} Th(C)$ ■

Observe that if in Theorem 2.7 we replace the sign "$\mathbb{K}_0$" by the sign "$SK_0$", where $SK_0 = \{C \in \mathbb{K}_0 : C$ is structural$\}$, and "correct" by "structurally correct", then the theorem remains true.

LEMMA 2.8. *Let* $0_t$ *be a characteristic consequence such that for every* $X \subseteq L$*,* $0_t(X) = X$*. Then* $card(\mathbb{K}_{0_t}) = 1$*.*

THEOREM 2.9. *If* $C \neq 0_t$ *(where* $0_t$ *is as in Lemma 2.8), then the following conditions are equivalent:*

*i)* $C$ *is compact,*

*ii)* $0_c$ *is compact,*

*iii)* $0_c$ *is finite.*

PROOF. i $\leftrightarrow$ ii is obvious.
ii $\to$ iii Assume that $0_c(X) = L$ and $X \neq L$. Thus $0_c(X_f) = L$ for some finite $X_f \subseteq X$ ($0_c$ is compact). Hence $0_c(X) = \bigcup\{0_c(X_f) : X_f \subseteq X$ and

$card(X_f) < \aleph_0\}$.
iii $\to$ ii Let $0_c$ be a finite consequence and $0_c(X) = L$, $X \subseteq L$.

1) If $X \neq L$ and for all finite $X_f \subseteq X$, $0_c(X_f) \neq L$, then $L = 0_c(X) = \bigcup\{0_c(X_f) : X_f \subseteq X \text{ and } card(X_f) < \aleph_0\} = X$. This contradicts the assumption that $X \neq L$.
2) If $X = L$, then for some $Y \neq L$, $0_c(Y) = L$ (because $0_c \neq 0_t$). Further proof in this case is analogous to 1) ■

COROLLARY 2.10. *Every finite characteristic consequence is correct.*

PROOF. Let $\mathcal{F} \subseteq S(0)$ and let $0(X) \neq L$ for some $X \subseteq L$ and some finite characteristic consequence 0. We will show that $X \subseteq Y$ for some $L \neq Y$. By both Theorem 2.9 and Lindenbaum's theorem (see [BS]) there exists a set $Y \neq L$ such that $X \subseteq Y$, $0(Y) = Y$ and $0(Y \cup \{a\}) = L$ for every $a \in L \backslash Y$.
Observe that for every $\Upsilon \in \mathcal{F}$, $Y \in \Upsilon$. Hence $Y \in \bigcap \mathcal{F}$ and $\bigcap \mathcal{F} \in S(0)$

THEOREM 2.11. *If $C \neq 0_t$ (where $0_t$ is as in Lemma 2.8), then the following conditions are equivalent:*

*i) $C$ is compact and $N$-structural,*
*ii) $0_c$ is compact and $N$-structural,*
*iii) $0_c$ is standard.*

PROOF. i $\leftrightarrow$ ii is obvious.
ii $\to$ iii According to Theorem 2.9 it will be enough to show that $0_c$ is structural. Let $e \in End(L)$ and let $X \subseteq L$. If $0_c(X) = L$, then $0_c(eX) = L$ ($0_c$ is $N$-structural) and $e(0_c(X) \subseteq o_c(X))$. If $0_c(X) \neq L$, then $e(0_c(X)) = e(X) \subseteq 0_c(e(X))$. So $0_c$ is structural.
iii $\to$ ii We must prove that $0_c$ is $N$-structural (see Theorem 2.9). Observe that if $0_c$ is not $N$-structural, then for some $e \in End(L)$ and for some finite $X_f \subseteq L$, $e(X_f) = e(L)$. But this is impossible because $card(e(L)) \neq card(e(X_f))$■

## §3. Characteristic consequence and maximality

In this section we shall consider only such characteristic consequences 0 for which the set $S\mathbb{K}_0 = \{C \in \mathbb{K}_0 : C \text{ is structural}\}$ is non-empty.

Definition 3.1. A structural consequence $C$ is said to be *maximal* iff for every structural $C_1$, $C < C_1$ implies $C_1(\emptyset) = L$.

Definition 3.2. A characteristic consequence 0 has a *maximal characterization* iff for every characteristic consequence $0^+ > 0$, if for some $\Upsilon^+ \in S^0(0^+)$, for some $\Upsilon \in S^0(0)$ and for every $\Upsilon'' \in S^0(0)$, $\Upsilon'' \subseteq \Upsilon$ implies $\Upsilon^+ \subset \Upsilon''$, then $0^+(\emptyset) = L$.

Lemma 3.3. *Let $C$ be a maximal consequence and let $\mathbb{K}_0$ be the characteristic class of* 0. *Then if $C \in \mathbb{K}_0$, then $S\mathbb{K}_0$ is a complete lattice.*

Corollary 3.4. *Let $C_1$, $C_2$ be maximal consequences. Then*

$$C_1 = C_2 \textit{ iff } 0_{c_1} = 0_{c_2}.$$

Theorem 3.5. *The following conditions are equivalent:*

*i) there exists a maximal consequence in* $\mathbb{K}_0$,
*ii)* 0 *is structurally correct and has a maximal characterization.*

Proof. i → ii By Lemma 3.3 and the remark following Theorem 2.7, it will be enough to show that 0 has a maximal characterization. Observe that for every $\Upsilon \in S^0(0)$, $Th(C)$ (where $C$ is maximal and $C \in \mathbb{K}_0$). If for some $0^+ > 0$ there exists $\Upsilon^+ \in S^0(0^+)$ and $\Upsilon \in S^0(0)$ such that for every $\Upsilon'' \subseteq \Upsilon$, $\Upsilon^+ \subset \Upsilon''$, then $\Upsilon^+ \subset Th(C)$. Hence $C_{\Upsilon^+} > C$ and $C_{\Upsilon^+}(\emptyset) = L$ (where the $C_{\Upsilon^+} = \Upsilon^+$). This implies that $0^+(\emptyset) = L$ as well (because $C_{\Upsilon^+} \in \mathbb{K}_{0^+}$).
ii → i Let $C$ denote the greatest consequence from $S\mathbb{K}_0$. If for some structural $C^+$, $C^+ > C$, then $0_{c^+} > 0_c = 0$. Observe that $\Upsilon^+ = Th(C^+)$ and $\Upsilon = Th(C)$ fulfill the assumption of Definition 3.2 and since 0 has a maximal characterization, then $0^+(\emptyset) = L = C^+(\emptyset)$. This shows that $C$ is maximal ■

Theorem 3.6. *Let* 0 *be a finite characteristic consequence. Then there exists a maximal consequence $C^0$ such that for every $C \in S\mathbb{K}_0$ $C \leqslant C^0$.*

Proof. Let $\mathcal{Y}$ be the set of all such subsets $Y$ of $L$ that $0(Y) \neq L$ and for every $e \in EndL$, $e(Y) \subseteq Y$. By the assumption that 0 is finite and by Zorn's lemma we conclude that there exists a maximal (with respect to inclusion) set $Y^+$ in $\mathcal{Y}$.

Let $C^{+}(X) = C''(X \cup Y^{+})$, where $C''$ is the greatest consequence of the complete lattice $S\mathbb{K}_0$ ($C''$ does exist by Corollary 2.10 and Theorem 2.7). $C^{+}$ is structural, Post-complete (see [T]) and $C^{+}(\emptyset) \neq L$.

Now let $C^0$ be such a structural and structurally complete consequence (see [T]) that $C^0 \geqslant C^{+}$, $C^0$ is Post-complete and $C^0(\emptyset) = C^{+}(\emptyset)$. By Theorem 6 from [T] $C^0$ is maximal and for every $C \in S\mathbb{K}_0$ $C \leqslant C'' \leqslant C^0$. This concludes the proof ■

COROLLARY 3.7. (Dziobiak, unpublished). *For every compact consequence $C$ there exists a maximal consequence $C^0$ such that $C \leqslant C^0$.*

## References

[BS] D. J. Brown and R. Suszko, *Abstract logics*, **Dissertationes Mathematicae**, vol. 102 (1973).

[T] M. Tokarz, *Connections between some notions of completeness of structural propositional calculi*, **Studia Logica**, vol. 32 (1973).

[W] R. Wójcicki, **Investigations into methodology of sentential calculi (I)**, Warsaw (1971).

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