Bulletin of the **S**ection of **L**ogic
Volume 7/4 (1978), pp. 167–173

reedition 2011 [original edition, pp. 167–175]

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ON THE DEGREE OF INCOMPLETENESS OF MODAL LOGICS (ABSTRACT)

In the following we will use the well-known correspondence between modal logics and varieties of modal algebras in our investigation of the function which assigns to a modal logic its degree of incompleteness. A *modal algebra* is an algebra $A = (A, +, \cdot, ', ^\circ, 0, 1)$ where $(a, +, \cdot, ', 0, 1)$ is a Boolean algebra and $^\circ$ is a unary operation satisfying $1^\circ = 1$ and $(x \cdot y)^\circ = x^\circ \cdot y^\circ$; $^\circ$ is called a modal operator. A *variety* of algebras is a class of algebras closed under the operations of forming homomorphic images, subalgebras as and direct products, and if $\underline{K}$ is a class of algebras then $V(\underline{K})$ denotes the smallest variety containing $\underline{K}$. The variety of modal algebras is denoted by $\underline{M}$, the subvariety of $\underline{M}$ defined by the equation $x^\circ \cdot x = x^\circ$ by $\underline{MR}$ and the subvariety of $\underline{M}$ defined by the equation $x^{\circ^n} = x^{\circ^{n-1}}$, n a natural number, by $\underline{M}^n$. Here $x^{\circ^0} = x$, $x^{\circ^n} = (x^{\circ^{n-1}})^\circ$, n a natural number. We write $\Lambda(\underline{K})$ for the lattice of subvarieties of a variety $\underline{K}$. If $\underline{K}$, $\underline{K}'$ are varieties such that $\underline{K} \subsetneqq \underline{K}'$ but for no variety $\underline{K}''$ $\underline{K} \subsetneqq \underline{K}'' \subsetneqq \underline{K}'$ then we say that $\underline{K}'$ is a cover of $\underline{K}$. The smallest normal modal logic – containing the axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and closed under the inference rules of modus ponens, substitution and necessitation – will be denoted by K; the lattice of modal logics which are extensions of K by $\Lambda(K)$. There is an obvious translation which assigns to any modal formula φ an $\underline{M}$-polynomial $\widehat{\varphi}$. The mapping

$$^* : \Lambda(K) \to A(\underline{M})$$

defined by

$$L \to L^* = \{A \in \underline{M} | A \models \widehat{\varphi} = 1, \varphi \in L\}$$

establishes an anti-isomorphism. In particular, $T^* = \underline{MR}$ (where T is the modal logic axiomatized, relative to K, by $\Box p \to p$) and $S4^* = \underline{MR} \cap \underline{M}^2 =$

$\underline{MR}^2$.

If $F = (W, R)$ is a Kripke frame (i.e., W is a non-empty set and R is a binary relation on W) then F^+ will be used to denote the modal algebra of all subsets of W, the modal operator $^\circ$ being defined by $A^\circ = \{w \in W | \forall v \in W[(w, v) \in R \Rightarrow v \in A]\}$. A modal algebra isomorphic to one of this kind is called a *Kripke algebra*. If $\underline{K}$ is a class of modal algebras then $\underline{K}_K$ denotes the subclass of its Kripke algebras. It is a simple matter to verify that for any modal formula φ and Kripke frame F we have $F \models \varphi$ iff $F^+ \models \widehat{\varphi} = 1$. In accordance with the usual terminology for modal logics we call a variety $\underline{K} \subseteq \underline{M}$ *complete* iff $\underline{K} = V(\underline{K}_K)$. Clearly, for any modal logic $L \in \Lambda(K)$, L is complete with respect to the Kripke semantics iff L^* is a complete variety.

Definition. If $\underline{K} \in \Lambda(\underline{M})$, the *degree of incompleteness* of $\underline{K}$, denoted by $\delta(K)$ is

$$|\{\underline{K}' \in \Lambda(\underline{M}) | \underline{K}'_K = \underline{K}_K\}|.$$

Thus for any $\underline{K} \in \Lambda(\underline{M})$, $1 \leq \delta(\underline{K}) \leq 2^{\aleph_0}$. The definition is nothing but an algebraic reformulation of the notion of degree of incompleteness of a modal logic, as introduced in [6] by K. Fine. He presented an example (as S. K. Thomason did in [9]) of a modal logic having degree of incompleteness ≥ 2. We showed in [2] that every modal logic satisfying certain mild assumptions has degree of incompleteness $2^{\aleph_0}$. We will now describe the behavior of the function δ in full detail.

For this we need the notion of splitting algebra, dealt with extensively in [4].

Definition. Let $\underline{K} \subseteq \underline{M}$ be a variety. A finite subdirectly irreducible algebra $A \in \underline{K}$ is called a *splitting algebra* in $\underline{K}$ if there exists a variety $\underline{K}_A \in \Lambda(\underline{K})$ such that for every $\underline{K}' \in \Lambda(\underline{K})$ either $A \in \underline{K}'$ or $\underline{K}' \subseteq \underline{K}_A$. If A is a splitting algebra then the variety $\underline{K}_A$ is called a *splitting variety* and is denoted by $\underline{K}/A$.

It is easy to see that if a variety $\underline{K} \subseteq \underline{M}$ is generated by its finite members – i.e., if the corresponding modal logic has the finite model property – then every splitting of $\Lambda(\underline{K})$ is determined by some splitting algebra. Each splitting variety $\underline{K}/A$ is definable, relative to $\underline{K}$, by a single equation $\varepsilon_A = 1$. In [4] we showed that in $\underline{MR}^2$ every finite subdirectly irreducible algebra os splitting in $\underline{MR}^2$; this result generalizes to the setting of $\underline{M}^n$, for any natural number n, as shown by W. Rautenberg [8]. Some examples in [3] showed that in MR not every finite subdirectly irreducible algebra is splitting.

THEOREM 1. *The only splitting algebra in $\underline{MR}$ is the two element modal algebra $\underline{2} = \{0, 1\}$, with $0^\circ = 0$, $1^\circ = 1$.*

THEOREM 2. *An algebra $A \in \underline{M}$ is splitting in $\underline{M}$ iff A is finite, subdirectly irreducible and satisfies $0^{0^n} = 1$ for some natural number n.*

The smallest splitting algebra in $\underline{M}$ is the two element modal algebra $\underline{2}^+ = \{0, 1\}$, with $0^0 = 1^0 = 1$. The variety $\underline{M}/\underline{2}^+$ is defined by the equation $0^0 = 0$ and corresponds with the modal logic D axiomatized relative to K by $\Diamond T$. The variety $\underline{M}/\underline{2}^+$ is the smallest splitting variety, and apparently does not contain any algebras which are splitting in $\underline{M}$; $\underline{2}$ is the only splitting algebra in $\underline{M}/\underline{2}^+$. Note also that through the class of splitting algebra is rather restricted, it generates $\underline{M}$.

LEMMA 3. *Let $\underline{K} \subseteq \underline{M}$ be a variety satisfying the equation $0^{0^n} = 1$ for some natural number n. Then the finitely generated algebras in $\underline{K}$ are finite; i.e., $\underline{K}$ is locally finite.*

Using this lemma we obtain:

THEOREM 4. *Let $\{A_i | i \in I\}$ be a set of splitting algebras in $\underline{M}$. Then $\bigcap_{i \in I} \underline{M}/A_i$ is generated by its finite members.*

It follows that varieties of this form are complete.

THEOREM 5. *Let $\{A_i | i \in I\}$ be a set of splitting algebras. Then $\delta(\bigcap_{i \in I} \underline{M}/A_i) = 1$.*

PROOF. Let $\underline{K} = \bigcap_{i \in I} \underline{M}/A_i$, $\underline{K}' \in A(\underline{M})$ such that $\underline{K}_K = \underline{K}'_K$. Since $\underline{K}$ is complete, $\underline{K}' \supseteq \underline{K}$. Since A_i is finite, $i \in I$, it is a Kripke algebra, hence $A_i \notin \underline{K}'$, $i \in I$, thus $\underline{K}' \subseteq \bigcap_{i \in I} \underline{M}/A_i = \underline{K}$.

COROLLARY 6. *There are $2^{\aleph_0}$ varieties of modal algebras having degree of incompleteness 1.*

Apparently our conjecture in [1] and [2] that every modal logic which is a proper extension of K has degree of incompleteness $2^{\aleph_0}$ is false. We will now proceed to show, however, that the varieties mentioned in Theorem 5 are the only ones having degree of incompleteness $< 2^{\aleph_0}$.

A variety is called *tabular* if it is generated by a finite algebra. In [4] it was shown that in $\underline{MR}^2$ every tabular variety is covered by tabular varieties only, and only by finitely many.

THEOREM 7. *In* $\underline{MR}^3$ *the variety* $V(2)$ *is covered by* $2^{\aleph_0}$ *varieties.*

In terms of modal logics this theorem clams that there are $2^{\aleph_0}$ modal logics containing the axioms $\Box p \to p$ and $\Box^2 p \to \Box^3 p$, which are immediate predecessors of classical logic, axiomatized by $\Box p \leftrightarrow p$. In particular, these logics need not be tabular. A variety is called *pretabular* if all its proper subvarieties are tabular. A well-known result, proved by several authors (see [7], [5]), states that $\underline{MR}^2$ contains only five pretabular varieties.

COROLLARY 8. $\underline{MR}^3$ *contains* $2^{\aleph_0}$ *pretabular varieties.*

Note that this result and the previous one are is contradiction with the results claimed in [8], section 3.

Using the varieties produced in the proof of Theorem 7 we are able to prove:

THEOREM 9. *Let* $\underline{K} \in A(\underline{M})$ *be a non-trivial variety and* A *a finite subdirectly irreducible algebra which is not splitting, such that* $A \notin \underline{K}$ *but such that all other homomorphic images of subalgebras of* A *do belong to* $\underline{K}$. *Then* $\delta(\underline{K}) = 2^{\aleph_0}$ *and* $\underline{K}$ *has* $2^{\aleph_0}$ *covers in* $\Lambda(\underline{M})$.

Using Theorem 9 and Theorem 5 we infer:

COROLLARY 10. *If* $\underline{K} \in A(\underline{M})$ *is such that* $\underline{K}$ *is not an intersection of splitting varieties, then* $\delta(\underline{K}) = 2^{\aleph_0}$.

Thus, if $\underline{K} \in \Lambda(\underline{M})$, then $\delta(\underline{K}) = 1$ if $\underline{K}$ is an intersection of splitting varieties, otherwise $\delta(\underline{K}) = 2^{\aleph_0}$. The proofs of theorems provide somewhat sharper results. In order to formulate them the following definition is useful.

DEFINITION. For $\underline{K} \in \Lambda(\underline{M})$, $\underline{K}' \in \Lambda(\underline{K})$ let

$$\delta_{\underline{K}}(\underline{K}') = |\{\underline{K}'' \in \Lambda(\underline{K}) | K''_K = \underline{K}'_K\}|.$$

It follows from the constructions that

Corollary 11.

(i) $\delta_{\underline{M}/\underline{2}^+}(\underline{K}) = 2^{\aleph_0}$, *for every* $\underline{K} \in \Lambda(\underline{M}/\underline{2}^+)$, *such that* $\underline{K}$ *is nontrivial and* $\underline{K} \neq \underline{M}/\underline{2}^+$.

(ii) $\delta_{\underline{MR}}(K) = 2^{\aleph_0}$, *for every* $\underline{K} \in \Lambda(MR)$ *such that* $\underline{K}$ *is nontrivial and* $\underline{K} \neq \underline{MR}$.

Hence, every proper extension of the modal logic T has degree of incompleteness (relative to T) $2^{\aleph_0}$. Since in $\underline{M}^n$, n a natural number, every finite subdirectly irreducible algebra is splitting, and hence the $\underline{M}^n$ contain many varieties which are intersections of splitting varieties, the function $\delta_{\underline{M}^n}$ assumes the value 1 very often. Much more we do not know about the $\delta_{\underline{M}^n}$, $n \geqslant 3$. For example, if $\underline{K} \in \Lambda(\underline{MR}^n)$ is non-trivial and tabular, what is $\delta_{\underline{MR}^n}(\underline{K})$, $n \geqslant 3$?

A bit more can be said in case $n = 2$. Indeed, $\delta_{\underline{MR}^2}(\underline{K}) = 1$, for every tabular variety $\underline{K}$, and more generally, for every variety $\underline{K} \subseteq \underline{MR}^2/F_n^+$, where $F_n = (\{0, 1, \ldots, n-1\}, \leq)$. However, as Fine's example shows [6], there exists a $\underline{K} \in \Lambda(\underline{MR}^2)$ such that $\delta_{\underline{MR}^2}(\underline{K}) \geq 2$.

As a byproduct we obtain interesting results on the covering relation in $\Lambda(\underline{M})$.

Definition. If $\underline{K} \in \Lambda(\underline{M})$, $\underline{K}' \subseteq \Lambda(\underline{K})$, let $c_{\underline{K}}(\underline{K}') = |\{\underline{K}'' \in \Lambda(\underline{K})| \ \underline{K}''$ covers $\underline{K}'\}|$. We write $c(K)$ for $c_{\underline{M}}(\underline{K})$.

If $\underline{K} \in \Lambda(\underline{M})$ and $\underline{K}' \in \Lambda(\underline{K})$, $K' \neq \underline{K}$, are such that $\underline{K}$ is generated by its finite members or $\underline{K}'$ is finitely axiomatizable then $c_{\underline{K}}(\underline{K}') \geq 1$. In [3] we gave examples of varieties $\underline{K}, \underline{K}'$ such that $\underline{K}' \subsetneqq \underline{K}$ and $c_{\underline{K}}(\underline{K}') = 0$.

Theorem 13.

(i) Suppose $\underline{K} \in \Lambda(\underline{M})$, $\underline{K} \neq \underline{M}$, *is an intersection of splitting varieties. If* $\underline{m}$ *is the smallest cardinal number such that* $\underline{K} = \bigcap_{i \in I} \underline{M}/A_i$, $|I| = \underline{m}$, *for a set of splitting varieties* $\{\underline{M}/A_i | i \in I\}$, *then* $c(\underline{K}) = \underline{m}$. *Hence* $1 \leq m \leq \aleph_0$ *in this case.*

(ii) If not, then $c(\underline{K}) = 2^{\aleph_0}$ *if* $\underline{K}$ *is non-trivial;* $c(\underline{K}) = 2$ *if* $\underline{K}$ *is trivial.*

Theorem 14.

(i) *For every* $\underline{K} \in \Lambda(\underline{M}/2^+)$, $\underline{K}$ *nontrivial and* $\underline{K} \neq \underline{M}/\underline{2}^+$, $c_{\underline{M}/\underline{2}^+}(\underline{K}) = 2^{\aleph_0}$.

(ii) *For every* $\underline{K} \in \Lambda(\underline{MR})$, $\underline{K}$ *nontrivial,* $\underline{K} \neq \underline{MR}$, $c_{\underline{MR}}(K) = 2^{\aleph_0}$.

Corollary 15. *For* $\underline{K} \in \Lambda(\underline{M})$, $\underline{K}$ *non-trivial,* $\underline{K} \neq \underline{M}$

$$\delta(\underline{K}) = 1 \quad \textit{iff } c(\underline{K}) \leq \aleph_0$$

and

$$\delta(\underline{K}) = 2^{\aleph_0} \ \textit{iff } c(\underline{K}) = 2^{\aleph_0}.$$

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