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THE REPRESENTATION THEOREM FOR THE ALGEBRAS DETERMINED BY THE FRAGMENTS OF INFINITE-VALUED LOGIC OF ŁUKASIEWICZ

This is an abstract of the paper presented at the seminar held by Prof. Andrzej Wroński at the Jagiellonian University, Cracow.

In this paper we shall give a characterization of D-algebras in terms of lattice ordered abelian groups. To make this paper self-contained we shall recall some notations from [4]. The symbols $\rightarrow, \wedge, \vee, \sim$ serve as implication, conjunction, disjunction, and negation, respectively. By D we mean a set of connectives from the list above containing the implication connective $\rightarrow$. By a D-formula we mean a formula built up in a usual way from an infinite set of the propositional variables and connectives from D. By a D-identity we mean an expression of the form $\alpha = \beta$, where α, β are D-formulas. By a D-algebra we mean an algebra of type determined by the connectives from D satisfying all D-identities from the following list:

$I0$. $(x \rightarrow (y \rightarrow x)) \rightarrow z = z$
$I1$. $(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z)$
$I2$. $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$
$I3$. $(x \wedge y) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z)$
$I4$. $x \rightarrow (y \vee z) = (y \rightarrow z) \rightarrow (x \rightarrow z)$
$I5$. $\sim x \rightarrow \sim y = y \rightarrow x$

The symbol K_D denotes the variety of all D-algebras.

A lattice ordered abelian group is a structure $\mathcal{G} = \langle G, +, 0, \leqslant \rangle$ such that $\langle G, +, 0 \rangle$ is an abelian group, $\langle G, \leqslant \rangle$ is a lattice and for every $a, b, c \in G$, $a \leqslant b$ iff $a + c \leqslant b + c$. A set $P = \{a \in G : 0 \leqslant a\}$ is a positive cone of $\mathcal{G}$. For every $a, b \in G$, $a \leqslant b$ iff $b \subseteq a + P$. Let us note that:

(i) $\langle P, +, 0\rangle$ is a commutative semigroup satisfying the cancellation laws and having a neutral element 0.

(ii) $\langle P, +, 0\rangle$ is a lattice ordered semigroup with the smallest element 0 with respect to the natural ordering ($a \leqslant b$ iff $b \in a + P$).

It follows from the theorem of Birkhoff [1] that a semigroup is a positive cone of a lattice ordered abelian group iff it satisfies all the conditions mentioned above.

For every $a, b \in P$ there exists the residual $a \to b$ defined as the smallest element of the set $\{c \in P : a + c \geqslant b\}$. It is proved in Bosbach [2] that the class of positive cones of lattice ordered abelian groups can be viewed as the variety of all algebras $\mathcal{P} = \langle P, +, \to\rangle$ of type $\langle 2, 2\rangle$ satisfying the following identities:

$(c0)$ $x + (x \to y) = y + (y \to x)$
$(c1)$ $(x + y) \to z = y \to (x \to z)$
$(c2)$ $x \to (y + z) = y$

We shall show that, analogously, the class of ideals of positive cones of lattice ordered abelian groups can be characterized as a variety of the algebras $\langle A, \to, \wedge\rangle$ of type $\langle 2, 2\rangle$, where for every $a, b \in A$, $a \wedge b = sup(a, b)$. Also the class of principal ideals can be characterized as a variety of the algebras $\langle B, \to, \sim\rangle$ of type $\langle 2, 2\rangle$, where for every $a, b \in B$, $\sim ((a \to\sim b) \to\sim a) = sup(a, b)$ and $\sim (a \to a)$ is the greatest element of B.

Following Bosbach, [3], by a residuation groupoid we mean a subalgebra of a reduct $\langle P, \to\rangle$ of a positive cone $\langle P, +, \to\rangle$ of a lattice ordered abelian group.

Representation Theorem

(i) *(Bosbach [3])* $K_{\{\to\}}$ *is the variety of all residuation groupoids;*

(ii) $K_{\{\to,\wedge\}}$ *is the variety of all ideals of positive cones of lattices ordered abelian groups;*

(iii) $K_{\{\to,\sim\}}$ *is the variety of all principal ideals of positive cones of lattice ordered abelian groups.*

References

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