Bulletin of the **S**ection of **L**ogic
Volume 7/4 (1978), pp. 185–189
reedition 2011 [original edition, pp. 185–190]

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A NOTE ON INCOMPLETENESS OF MODAL LOGICS WITH RESPECT TO NEIGHBOURHOOD SEMANTICS

This is a summary of a lecture read at the Seminar of the Department of Mathematical Logic held by Professor Jerzy Kotas, Institute of Mathematics, N. Copernicus University, Toruń, June 1978.

§0. By a modal logic we understand a proper subset of the set of propositional modal formulae that contains all classical tautologies, the axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and closed under modus ponens, substitution and necessitation. In our considerations all neighbourhood frames are normal, i.e. such that neighbourhoods of each point constitute a filter. For a neighbourhood frame $\underline{F} = (\underline{U}, N)$, by $\underline{F}^+$ we denote the algebra $(P(U), \cup, \cap, \neg, \Box_N, 0, 1)$; where $(P(U), \cup, \cap, \neg, 0, 1)$ is the well known Boolean algebra and $\Box_N$ is a unary operation defined as follows: $\Box_N(S) = \{x \in U | S \in N(x)\}$ for every $S \subseteq U$. We know that $E(\underline{F}) = E(\underline{F}^+)$, where $E(\underline{F})$ $(E(\underline{F}^+))$ is the set of all formulae which are valid in $\underline{F}$ $(\underline{F}^+)$. Following Fine [3], for a modal logic L, we put $\delta^*(L) = card\{L' | L'$ is a modal logic such that for every neighbourhood frame $\underline{F}$, $L' \subseteq E(\underline{F})$ iff $L \subseteq E(\underline{F})$. Our aim is to prove the following theorem which is a counterpart of Blok's one (see [1], also [2]).

Theorem 1. *For any modal logic L:*

i) if $\Box p \to \Diamond p \in L$ and $\Box^n p \to \Box^{n+1} p \in L$ for some $n \geqslant 0$, then $\delta^*(L) = 2^{\aleph_0}$

ii) if $p \to p \in L$, then $\delta^*(L) = 2^{\aleph_0}$.

§1. In order to prove the first part of Theorem 1 let us take into consideration the formulae

$\alpha_{n,k} = (p \wedge \Diamond^{2n} q) \to (\Diamond^n q \vee \Diamond^{2n}(q \wedge \Diamond^{k(n+1)} p));\ n \geqslant 1, k \geqslant 1$

$\beta_n = (\Box^n p \wedge \sim \Box^{n+1} p \wedge \sim \Box^{2n+1} p) \to \Diamond^n (\Box^{2n+1} p \wedge \sim \Box^{2n+2} p \wedge \sim \Box^{3n+2} p),$
$n \geqslant 1$

$$\gamma_n = (\Box^n p \wedge \sim \Box^{n+1} p \wedge \sim \Box^{2n+1} p) \to \sim (r \bigwedge_{1 \leqslant i \leqslant 2n+4} \Box^n (q_i \to r) \wedge \wedge \bigwedge_{1 \leqslant i \leqslant 2n+4} \Box^n (r \to \Diamond^n q_i) \wedge \bigwedge_{1 \leqslant i \neq j \leqslant 2n+4} \Box^n \sim (q_i \wedge q_j)),\ n \geqslant 1.$$

These formulae can be found in Thomason [5].

LEMMA 1. *For any $n \geqslant 1$ and every neighbourhood frame $\underline{F}$:*
if $\alpha_{n,k} (k \geqslant 1) \beta_n, \gamma_n \in E(\underline{F})$, then
$\Box^n p \to (\Box^{n+1} p \vee \Box^{2n+1} p) \in E(\underline{F})$.

In the proof the ideas from Gerson [4] are used. Let us recall Blok's definition of a family of modal algebras $A_{\underline{m}}$ (cf. [1]). $A_{\underline{m}}$ is a modal algebra of finite and cofinite subsets of the set of natural numbers N. The operation $\Box_{\underline{m}}$ in $A_{\underline{m}}$ corresponding to a connective $\Box$ is defined as follows: for $M \subseteq N$

$$\Box_{\underline{m}} = \begin{cases} \emptyset, & \text{if } M \text{ is finite} \\ [m_{i+1}, \infty), & \text{if } N \neq M \text{ is cofinite and } i = min\{j | [m_j, \infty) \subseteq M\} \\ N, & \text{if } M = N \end{cases}$$

where $\underline{m} = (m_i)_{i=1}^{\infty}$ is a sequence of natural numbers satisfying $m_1 = 3$, $m_2 = 4$, $m_{i+1} > m_i$ and $m_{i+1} - m_i \leqslant 2$, for $i \geqslant 1$.

LEMMA 2. *For each algebra $A_{\underline{m}}$*
$\alpha_{n,k}, \beta_n, \gamma_n \in E(A_{\underline{m}})\ (n \geqslant 1, k \geqslant 1)$.

For any class $\underline{K}$ of algebras, $V(\underline{K})$ denotes the smallest variety that contains $\underline{K}$, and $V(\underline{K})_{SI}$ is the class of all subdirectly irreducible members of $V(\underline{K})$. The next lemma is an immediate consequence of Theorem 4.4 in Blok [1].

LEMMA 3. *For every algebra $A_{\underline{m}}$ and natural number $n \geqslant 1$:*
if $\underline{B} \in V(A_{\underline{m}})_{SI}$ and $\Box^n p \to (\Box^{n+1} p \vee \Box^{2n+1} p) \in E(\underline{B})$, then $\underline{B} \cong \underline{2}$.

Let L be a modal logic. By $\underline{K}_L$ we denote the variety of modal algebras corresponding with L. If $\Box p \to \Diamond p \in L$, then $\underline{2} \in \underline{K}_L$. Hence, from the Birkhoff theorem and the above lemmas we obtain:

COROLLARY 1. *Let L be a modal logic such that $\Box p \to \Diamond p \in L$ and for some $n \geqslant 1$ the formulas $\alpha_{n,k}$ $(k \geqslant 1)$, β_n and γ_n are theses of L. Then, for every algebra $A_{\underline{m}}$ and neighbourhood frame F, $\underline{F}^+ \in V(\underline{K}_L \cup \{A_{\underline{m}}\})$ iff $\underline{F}^+ \in \underline{K}_L$.*

Let us suppose $\Box p \to \Diamond p \in L$ and $\Box^n p \to \Box^{n+1} p \in L$, for some $n \geqslant 0$. Blok [1] has proved that $V(\underline{K}_L \cup \{A_{\underline{m}}\}) \neq V(\underline{K}_L \cup \{A_{\underline{n}}\})$, for every $\underline{m} \neq \underline{n}$. But L contains also the formulae $\alpha_{n+1,k}$ $(k \geqslant 1)$, β_{n+1}, and γ_{n+1}, and so, by Corollary 1, we receive $\delta^*(L) = 2^{\aleph_0}$.

§2. Now, similarly, we prove the second part of Theorem 1. Therefore take the following formulae:

$$\alpha_{n,k} = (p \wedge \Diamond^4 q) \to ((\Box^{n+1} r \to \Box^n r) \vee \Diamond^2 q \vee \Diamond^4 (q \wedge \Diamond^k p));$$
$$n \geqslant 0, k \geqslant 3$$

$$\beta = (\Box p \wedge \sim p) \to \Diamond^2 (\Box^2 p \wedge \sim \Box p)$$

$$\gamma = (\Box p \wedge \sim p) \to \sim (r \wedge \bigwedge_{1 \leqslant i \leqslant 5} \Box^2 (r \to \Diamond^2 q_i) \wedge \bigwedge_{1 \leqslant i \leqslant 5} \Box^2 (q_i \to r) \wedge$$
$$\wedge \bigwedge_{1 \leqslant i \neq j \leqslant 5} \Box^2 \sim (q_i \wedge q_j))$$

These formulae will play a similar role to that in the previous section. Indeed, for them we have (cf. Lemma 1).

LEMMA 4. *For a neighbourhood frame $\underline{F}$:*
if $\alpha_{n,k}$ $(n \geqslant 0, k \geqslant 3), \beta, \gamma \in E(\underline{F})$, then $\Box p \to p \in E(\underline{F})$.

Let b_i, $i = 1, 2, 3, 4, 5$, be arbitrary but fixed elements not belonging to the set of natural numbers N, and let $(a_n)_{n=1}^{\infty}$ be a one-to-one sequence of such elements. For any sequence $\underline{m} = (m_i)_{i=1}^{\infty}$ of natural numbers such that $m_1 = 2$, $m_i < m_{i+1}$ and $m_{i+1} - m_i \leqslant 2$ $(i \geqslant 1)$, let us put $W_{\underline{m}} = N \cup \{b_1, b_2, b_3, b_4, b_5\} \cup \{a_n | n \in \underline{m}\}$ and $R_{\underline{m}} = \{(b_i, b_i) | i \in \{1, 2, 3, 4, 5\}\} \cup \{(b_i, b_{i+1}, (b_{i+1}, b_i) | i \in \{1, 2, 3\}\} \cup \{(b_i, 1) | i \in \{1, 2, 3, 4\}\} \cup \{(1, b_4), (b_1, b_5), (b_5, b_4), (b_1, b_4), (1, 1)\} \cup \{(n, m) | n < m\} \cup \{(n+1), n) | n \geqslant 1\} \cup \{(n, a_m) | n \leqslant m \wedge m \in \underline{m}\} \cup \{(a_n, n), (a_n, a_n) | n \in \underline{m}\}$. Given $(W_{\underline{m}}, R_{\underline{m}})$, let $B_{\underline{m}}$ denote the modal algebra of finite and cofinite subsets of $W_{\underline{m}}$ in which the operation $\Box_{\underline{m}}$ corresponding to the connective $\Box$ is defined with the aid of $R_{\underline{m}}$, i.e. $\Box_{\underline{m}}(S) = \{x \in W_{\underline{m}} | \forall y (x R_m y \Rightarrow y \in S)\}$.

LEMMA 5. *For each algebra* $B_{\underline{m}}$

i) $\alpha_{n,k}, \beta, \gamma \in E(B_{\underline{m}})$ $(n \geqslant 0, k \geqslant 3)$

ii) if $\underline{B} \in V(B_{\underline{m}})_{SI}$ *and* $\Box p \to p \in E(\underline{B})$, *then* $\underline{B} \cong \underline{2}$.

Lemma 4 and 5 allow us to obtain the following

COROLLARY 2. *Let* L *be a modal logic such that* $\Box p \to p \in L$. *Then, for every algebras* $B_{\underline{m}}$ *and neighbourhood frame* $\underline{F}$, $\underline{F}^+ \in V(\underline{K}_L \cup \{B_{\underline{m}}\})$ *iff* $\underline{F}^+ \in \underline{K}_L$.

Each of the algebras $B_{\underline{m}}$ is subdirectly irreducible and $\Box p \to p \notin E(B_{\underline{m}})$. Applying the method due to Blok [1] one can prove.

LEMMA 6. *For a modal logic* L:

if $\Box p \to p \in L$ *then* $V(\underline{K}_L \cup \{B_{\underline{m}}\}) \neq V(\underline{K}_L \cup \{B_{\underline{n}}\})$ *for every* $\underline{m} \neq \underline{n}$.

Corollary 2 and Lemma 6 yield the second part of Theorem 1.

§3. We say that a modal logic L is complete with respect to neighbourhood semantics iff $L = \bigcap\{E(\underline{F}^+) | \underline{F}^+ \in \underline{K}_L\}$. We can neither prove nor disprove the following statement (comp. Lemma 4.1 [1]): L is complete with respect to neighbourhood semantics iff $\underline{K}_L = V(\{\underline{F}^+ \in \underline{K}_L | \underline{F}^+$ is subdirectly irreducible$\})$. If it were proved true, then Theorem 1 would follow immediately from Blok [1].

References

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