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TOPOLOGIES ON QUANTUM LOGICS INDUCED BY THE SET OF STATES

The term 'quantum logic' has several meanings. Sometimes it refers to non-classical pure propositional (see [3]) or predicate calculus (see [1]), but more often it is used as the name of a wide class of relational-algebraic structures, like CROSS (see [5]), partial algebras (see [4]) or orthomodulary posets (see [2]). We adopt here this last sense, and our definitions and notations will be roughly the same as in [2]. Just recall that a quantum logic is a set $\underline{L}$ of propositions, endowed with a partial order and an involutory orthocomplementation, whose all elements are bounded by 0 (below) and I (above), plus a set $\underline{S}$ of states which we interpretate as non-classical probability measures on the set of propositions. This means that the pair $(\underline{L}, \underline{S})$ characterizes our quantum logic.

We aim to construct a topology on $\underline{L}$, induced in a more or less natural way, by the usual topology on the closed $\mathbb{R}$-segment $[0, 1]$. Actually, we are looking for the weakest topology for which all the states $\underline{s}$ in $\underline{S}$

$$\underline{L} \longrightarrow^{s} [0, 1]$$

are continuous.

Consider the set of all sets of the following form:

$\underline{s}^{-1}(I)$, for every state $\underline{s}$ in $\underline{S}$, and any open interval included in $[0, 1]$. The class B of all finite intersections of such sets is the basis for a topology on the set $\underline{L}$. (We are assuming, of course, that there are "many" states in $\underline{S}$).

Given a fixed "point" (proposition) p in $\underline{L}$, a typical neighbourhood of p is a set like $s_1^{-1}(I_1) \cap \ldots \cap s_n^{-1}(I_n)$, for a suitable collection of states $\underline{s}_i$ and open intervals $\underline{I}_i$. This topology on $\underline{L}$ is labelled T and as usual, the topological space resulting from endowing $\underline{L}$ with the topology $\underline{T}$ – is denoted by $(\underline{L}, \underline{T})$.

Recall the standard notation (for instance, [2], p. 82): for a proposition $\underline{p}$, the proposition $\underline{p}'$ is its orthocomplement. We write $p \circ q$ for "p and q are disjoint", usually defined as $p \leqslant q'$ or (equivalently) $q \leqslant p'$.

Proposition 1. *Define the function $f : \underline{L} \to \underline{L}$ as $f(p) = p'$. Then f is an auto-homomorphism in $(\underline{L}, \underline{T})$.*

Proof. Obviously, for any p in $\underline{L}$, we have $f(f(p)) = p$. Then f is not just a bijection but an involutory one; i.e., $f^{-1} = f$.

Let $\underline{U}$ be a member of a sub-base for $(\underline{L}, \underline{T})$. Thus, there is a state $\underline{s}$ in $\underline{S}$, and an open subinterval $\underline{I}$ of $[0, 1]$, such that $\underline{U} = \underline{s}^{-1}(\underline{I})$. Apply $\underline{f}$ to both members:

$$\begin{aligned} \underline{f}(U) &= \underline{f}(\underline{s}^{-1}(\underline{I})) \\ &= \underline{f}(\{p \in L/\underline{s}(p) \in \underline{I}\}) \\ &= \{p' \in L/\underline{s}(p) \in I\} \\ &= \{p'' \in L/\underline{s}(p') \in \underline{I}\} \\ &= \{p \in L/1 - \underline{s}(p) \in \underline{I}\} \end{aligned}$$

The last equality was inferred by the rules $p'' = p$ and $\underline{s}(p') = 1 - \underline{s}(p)$. Now, for each $\underline{x}$ in $\underline{I}$, we write

$$\begin{aligned} \underline{A}(\underline{x}) &= \{p \in \underline{L}/1 - \underline{s}(p) = x\} \\ &= \{p \in \underline{L}/\underline{s}(p) = 1 - x\} \end{aligned}$$

We have that $\bigcup \underline{A}(\underline{x}) = \underline{d}(U)$

$$AND \bigcup_{x \in I} \underline{A}(\underline{x}) = \{p \in \underline{L}/s(p) \in (1 - \underline{I})\}$$

where $(1 - \underline{I}) =_{df.} \{1 - \underline{x}/0 < \underline{x} < 1\}$, is an open interval contained in $[0, 1]$.

Thus, $f(U) = s^{-1}(1 - I)$. We conclude that $f(U)$ is a T-open set, and $\underline{f}$ is an open mapping. Moreover, the relation $f = f^{-1}$ entails that $\underline{f}$ is also continous and closed.

General Assumption. From now on we assume that $\underline{S}$ has denumerably many states, but still separates the "points" of $\underline{L}$ (see [2], p. 82, def. of "*full*" logic), and also "points" from closed sets.

Proposition 2. *Under our general assumption, (L, T) is a metrizable space.*

PROOF. (i) Since $\underline{S}$ is denumerable, $(\underline{L}, \underline{T})$ satisfies the second axiom of denumerability. Just pick the open intervals in $[0, 1]$ which have rational end points.

(ii) $(\underline{L}, \underline{T})$ is Hausdorff. Assume $p \neq q$, two propositions of $\underline{L}$. Then there is a state $\underline{s}$, such that $\underline{s}(p) \neq \underline{s}(q)$, because of the "fullness". Choose two open intervals, $\underline{J}$ and $\underline{I}$ in $[0, 1]$, such that $\underline{s}(p) \in \underline{J}$ and $s(q) \in \underline{I}$, and $\underline{I}$ and $\underline{J}$ are disjoint. Hence, their inverse images in $\underline{L}$ are disjoint neighbourhoods of $\underline{p}$ and $\underline{q}$, respectively.

(iii) Since $\underline{S}$ separates elements in L, we can consider the following construction which depends upon denumerability of $\underline{S}$

$$\underline{L} \longrightarrow^{s_j} Q_j$$

where $j \in \omega$, and $Q_j = [0, 1]$ for every non-negative integer j.

Because of the very construction of our topology $\underline{T}$ in L, we can embed $\underline{L}$ in the Hilbert's cube $[0, 1]^\omega$. On applying the so called Urysohn's lemma, (L, T) becomes metrizable (and also separable).

Consider now $(\underline{L}, \underline{T})$ as a topological space with partial operations of "supremum" and "infimum", and $\underline{S}$ as a very "sui generis" set of functionals into the unitary real interval.

PROPOSITION 3. *Under the conditions above, the partial operations*

$$g : (p, q) \to (p \vee q)$$

$$h : (p, q) \to (p \wedge q)$$

are <u>*weakly*</u> *continuous.*

PROOF. Let $\underline{A}$ be the following subset of $\underline{L} \times \underline{L}$,

$$\underline{A} = \{(p, q) \in \underline{L} \times \underline{L} / p \, \phi \, q\}$$

We take $\underline{A}$ as a domain of definition for q and h.

(i) Let $(p_n, q_n) \to (p, q)$, where the convergence of sequences of propositions is in the (product) topology $\underline{T}$ of $\underline{L}$. Any state $\underline{s}$ is continuous by the very definition of the T-topology; thus

$$\underline{s}(p_n) \rightarrowtail \underline{s}(p)$$

$$\underline{s}(q_n) \rightarrowtail \underline{s}(q)$$

in the usual topology of real numbers. By the continuity of addition in $\mathbb{R}$, we get

$$\underline{s}(p) + \underline{s}(q) = LIM\,\underline{s}(p_n) + \underline{s}(q_n)$$

But from definition of $\underline{s}$ and $\underline{A}$

$$\underline{s}(p \vee q) = LIM\,\underline{s}(p_n \vee q_n)$$

Since this equality holds for every *state*, we have here something equivalent to the weak convergence of

$$(p_n \vee q_n) \longrightarrow^{\text{for } s \in S} (p \vee q).$$

Therefore $\underline{v}$ is a weak-S continuous partial operation (on A).

(ii) Just write $\underline{h}$ in terms of $\underline{g}$ and $\underline{f}$. I.e., for every

$$(p, q) \in A,\ \ h(p, q) = (p \wedge q) = (p' \vee q')',$$

and the result follows from S-continuity of $\underline{v}$ and strong continuity of f.

Remark. Recall that any state $\underline{s}$ is additive on disjoint propositions.

In the general case, we assume that the states are additive, (actually, σ-additive) on disjoint sequences of propositions. This by no means, implies that the states of $\underline{S}$ are additive *only* on disjoints propositions. We call $\underline{S}'$ the subset of $\underline{S}$ such that:

"For every $\underline{s} \in \underline{S}'$, if $\underline{s}(p_l \vee \ldots \vee p_n \vee \ldots) =$
$= \underline{s}(p_1) + \ldots + \underline{s}(p_n) + \ldots$, then, for $i \neq j$ $p_i \between p_j$"

Proposition 4. *The set $\underline{A}$ of mutually disjoint propositions in $\underline{L} \times \underline{L}$ is weakly-S' closed. I.e., if a sequence (p_n, q_n) of elements from $\underline{A}$ converges to (p, q) in the sense of S'-functionals, then (p, q) is in $\underline{A}$.*

Proof. Assume the following relations of convergence:

$p_n \rightarrowtail p$ and $q_n \rightarrowtail q$ in the weak-S'. This means that for every $\underline{s}$ in $\underline{S}'$

$\underline{s}(p_n) \rightarrowtail s(p)$ and $\underline{s}(q_n) \rightarrowtail s(q)$ in the topology of the unitary real interval. Using continuity of + in the real line:

(+) $\underline{s}(p_n) + \underline{s}(q_n) \rightarrowtail \underline{s}(p) + \underline{s}(q)$

As p_n and q_n are in A, we have $\underline{s}(p_n) + \underline{s}(q_n) = s(p_n \vee q_n)$.
In turn, due to continuity of "$\vee$", we get:

$$\underline{s}(p_n \vee q_n) \rightarrowtail \underline{s}(p \vee q) \quad \text{or}$$

(++) $\underline{s}(p_n) + \underline{s}(q_n) \rightarrowtail \underline{s}(p \vee q)$.
Due to uniqueness of the limit:

$$\underline{s}(p \vee q) = \underline{s}(p) + \underline{s}(q) \text{ using } (+) \text{ and } (++)$$

But by applying the very definition of $\underline{S}'$ we obtain:

$$p \between q \text{ what means } (p, q) \in A.$$

Final Remarks. This is still a programatical attempt to find a way for a "reasonable" topologization of the orthoposet of propositions in quantum logic. It is superfluous to point out that our results are still very crude, and lay heavily on not totally natural assumptions, like in the case of denumerability of S. Moreover, some "smoth" connection with the topologies recently introduced into the state-space, will be desirable. These problems should wait for future research. Specifically, we would like to make a more profound analysis of some new proposals in this field, e.g. [6].

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