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A PURELY ALGEBRAIC PROOF OF THE OMITTING TYPES THEOREM

In the present note we make use of some information given in [2]. Also, the terminology and notation do not differ from those accepted in [2]; in particular this concerns the formalism for the predicate calculus.

Let $\mathcal{A}$ be a model of a first-order language $\mathcal{L}$. We say that $\mathcal{A}$ realizes a set of formulas $\Sigma \subseteq Fla(\mathcal{L})$ iff $\mathcal{A} \models \sigma[\bar{a}]$ for some valuation $\bar{a}$ in $\mathcal{A}$ and all $\sigma \in \Sigma$. We say that $\mathcal{A}$ omits Σ iff $\mathcal{A}$ does not realize Σ.

A formula $\alpha \in Fla(\mathcal{L})$ is consistent with a theory T in iff there is a model of T which realizes $\{\alpha\}$.

Lemma 1. *Let T be a theory in a countable language $\mathcal{L}$. Let $\alpha \in Fla(\mathcal{L})$. The following conditions are equivalent:*

(a) *α is consistent with T*

(b) *$|\alpha|_T \neq 0_T$ in the Lindenbaum-Tarski algebra $\mathcal{B}_T$.*

Lemma 1 is a direct consequence of Corollary 3 in [2]. See also [1, p. 105].

Let $\Sigma \subseteq Fla(\mathcal{L})$. A theory T in $\mathcal{L}$ is said to locally realize Σ iff there is a formula φ in $\mathcal{L}$ such that

(i) φ is consistent with T

(ii) for all $\sigma \in \Sigma$, $T \models \varphi \to \sigma$

T locally omits Σ iff T does not locally realize Σ.

If $\Sigma \subseteq Fla(\mathcal{L})$ and $\varepsilon \in Sb(\mathcal{L})$, then $\varepsilon\Sigma =_{df} \{\varepsilon\sigma : \sigma \in \Sigma\}$.

We say that Σ is a set of formulas of $\mathcal{L}$ in the free individual variables $x_1, \ldots, x_n$ (symbolically $\Sigma = \Sigma(x_1, \ldots, x_n)$) iff $x_1, \ldots, x_n$ are distinct free individual variables and every formula σ in Σ contains at most the variables $x_1, \ldots, x_n$.

Lemma 2. *Let T be a theory in a language $\mathcal{L}$. Suppose T locally omits $\Sigma = \Sigma(x_1, \ldots, x_n)$. Then T locally omits $\varepsilon\Sigma$, for all $\varepsilon \in Sb(\mathcal{L})$.*

Lemma 3. *Let T be a theory in a countable language $\mathcal{L}$. The following conditions are equivalent:*

(i) *T locally omits $\Sigma = \Sigma(x_1, \ldots, x_n)$*

(ii) $\bigcap_{\sigma \in \Sigma} |\sigma|_T = 0_T$ *in $\mathcal{B}_T$.*

Corollary 4. *If a theory T in a countable language locally omits $\Sigma = \Sigma(x_1, \ldots, x_n)$, then*

$(\Sigma)_\varepsilon \qquad \bigcap_{\sigma \in \Sigma} |\varepsilon\sigma|_T = 0_T$

for all $\varepsilon \in Sb(\mathcal{L})$.

Notice that if $\Sigma = \Sigma(x_1, \ldots, x_n)$ and $\mathcal{L}$ is countable, then the family of all meets $(\Sigma)_\varepsilon$, $\varepsilon \in Sb(\mathcal{L})$, is at most countable.

Theorem (Extended Omitting Types Theorem) [1, p. 82]. *Let T be a consistent theory in a countable language $\mathcal{L}$, and for each $k \in N$ let $\Sigma_k = \Sigma_k(x_1, \ldots, x_{n_k})$ be a set of formulas in n_k free individual variables. If T locally omits each Σ_k, then T has a countable model which omits each Σ_k.*

Proof. From assumption and Corollary 4 we obtain:

$(\Sigma_k)_\varepsilon \qquad \bigcap_{\sigma \in \Sigma_k} |\varepsilon\sigma|_T = 0_T$ in $\mathcal{B}_T$

for each $k \in N$ and every substitution $\varepsilon \in Sb(\mathcal{L})$. The family of all meets $(\Sigma_k)_\varepsilon$, $k \in N$, $\varepsilon \in Sb(\mathcal{L})$, is also countable. By Rasiowa-Sikorski Lemma (Lemma 2 in [2]) there exists a Q-ultrafilter ∇ in $\mathcal{B}_T$ which preserves all the meets $(\Sigma_k)_\varepsilon$, $k \in N$, $\varepsilon \in Sb(\mathcal{L})$. It follows that for every $\varepsilon \in Sb(\mathcal{L})$ and each $k \in N$ there is a formula in Σ_k, say $\sigma_{\varepsilon,k}$, such that $|\neg\varepsilon\sigma_{\varepsilon,k}|_T \in \nabla$. Let $\mathcal{A}_\nabla$ be the algebraic model determined by ∇. Suppose that for some $k_0 \in N$ the model $\mathcal{A}_\nabla$ realizes Σ_{k_0}. Then, due to formula (TL) in [2], there exists an $\varepsilon_0 \in Sb(\mathcal{L})$ such that $|\varepsilon_0\sigma|_T \in \nabla$ for all $\sigma \in \Sigma_{k_0}$. In particular $|\varepsilon_0\sigma_{\varepsilon_0,k_0}|_T \in \nabla$. Contradiction.

References

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[3] H. Rasiowa and R. Sikorski, **The mathematics of metamathematics**, second edition revised, PWN, Warszawa 1968.

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