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A NOTE ON $R - S$ LEMMA

Rauszer and Sabalski proved in [2] that distributivity with respect to infinite joins and meets is a sufficient and necessary condition making the Rasiowa-Sikorski Lemma valid in distributive lattices. The main part of their proof is a direct construction of a required filter under distributivity. In this note we show that a generalization of the result can be obtained from the Rasiowa-Sikorski Lemma for Boolean algebras by using Görnemann's result in [1] instead of a direct construction.

Suppose A is a distributive lattice and $Q, R \subseteq 2^A - \{\emptyset\}$. We call A (Q, R) complete if $\forall M \in Q \exists \bigsqcap M \in A$ and $\forall N \in R \exists \bigsqcup N \in A$. $M \in Q$ is $\bigsqcap$-dis if $\exists \bigsqcap M \in A$ and $\forall a \in A\ a \sqcup \bigsqcap M = \bigsqcap_{m \in M}(a \sqcup m)$. $N \in R$ is $\bigsqcup$-dis if $\exists \bigsqcup N \in A$ and $\forall a \in A\ a \sqcap \bigsqcup N = \bigsqcup_{n \in N}(a \sqcap n)$. (Q, R) is called distributive if every $M \in Q$ is $\bigsqcap$-dis and every $N \in R$ is $\bigsqcup$-dis.

Suppose A is (Q, R) complete and $C, D \subseteq A$. By $\nabla_C(\Delta_D)$ we mean the filter (ideal) in A generated by $C(D)$, in particular, let $\nabla_\emptyset = \Delta_\emptyset = \emptyset$. (C, D) is called (Q, R) complete if (i) and (ii), where

(i) $\forall M \in Q(\nabla_C \cap \Delta_{D \cup \{\bigsqcap M\}} = \emptyset \Rightarrow \exists m \in M \nabla_C \cap \Delta_{D \cup \{m\}} = \emptyset)$

(ii) $\forall N \in R(\nabla_{C \cup \{\bigsqcup N\}} \cap \Delta_D = \emptyset \Rightarrow \exists n \in N \nabla_{C \cup \{n\}} \cap \Delta_D = \emptyset)$

Remark. We call a filter $F \subseteq A$ (an ideal $I \subseteq A$) $Q(R)$ complete if $\forall M \in Q$ $M \subseteq F \Rightarrow \bigsqcap M \in F$ ($\forall N \in R\ N \subseteq I \Rightarrow \bigsqcup N \in I$). Then, ∇_C is Q complete iff $(C, \emptyset)$ is (Q, R) complete and Δ_D is R complete iff $(\emptyset, D)$ is (Q, R) complete.

For $a, b \in A$, $a \leqslant b\ mod(C, D)$ is defined by $\nabla_{C \cup \{a\}} \cap \Delta_{D \cup \{b\}} \neq \emptyset$. An equivalence relation $\sim$ is then defined as $a \sim b$ if $a \leqslant b\ mod(C, D)$ and $b \leqslant a\ mod(C, D)$. We denote the equivalence class of $a \in A$ by $[a]$. $A/\sim$ becomes a distributive lattice with the order defined as $[a] \sqsubseteq [b]$ if $a \leqslant b\ mod(C, D)$. We denote the lattice by $A/(C, D)$. Clearly, $[a \sqcap b] = [a] \sqcap [b]$ and $[a \sqcup b] = [a] \sqcup [b]$ for any $a, b \in A$.

For $Q \subseteq 2^A - \{\emptyset\}$, let $[Q] = \{[M] : [M] = \{[m] : m \in M\}$ and $M \in Q\}$.

LEMMA 1. *Suppose A is a (Q,R) complete distributive lattice and $C, D \subseteq A$. Then, the following are equivalent.*

(a) *For any $a, b \in A$, $(C \cup \{a\}, D \cup \{b\})$ is (Q,R) complete.*

(b) *$A/(C,D)$ is a $([Q],[R])$ complete distributive lattice with $\bigsqcap[M] = [\bigsqcap M]$ and $[\bigsqcup N] = \bigsqcup[N]$ for every $M \in Q$ and for every $N \in R$, and $([Q],[R])$ is distributive.*

PROOF. (a) $\Rightarrow$ (b). First, we show that $\bigsqcap_{m\in M}[b \sqcup m] = [b \cup \bigsqcap M]$ for any $b \in A$ and for any $M \in Q$. It is trivial that $[b \sqcup \bigsqcap M] \subseteq [b \sqcup m]$ for any $m \in M$. Suppose $[a] \sqsubseteq [b \sqcup m]$ for any $m \in M$. Then, for any $m \in M$, $a \leqslant b \bigsqcup m \ mod(C,D)$, namely $\nabla_{C\cup\{a\}} \cap \Delta_{D\cup\{b,m\}} \neq \emptyset$. By (a), $\nabla_{C\cup\{a\}} \cap \Delta_{D\cup\{b,\bigsqcap M\}} \neq \emptyset$. So, $a \leqslant b \sqcup \bigsqcap M \ mod(C,D)$ and the $[a] \sqsubseteq [b \sqcup \bigsqcap M]$. Thus, $\bigsqcap_{m\in M}[b \sqcup m] = [b \sqcup \bigsqcap M]$. Now, take $\bigsqcap M$ for b and we have $\bigsqcap[M] = [\bigsqcap M]$ since $\bigsqcap M \sqcup m = m$ for any $m \in M$. Hence, $\bigsqcap_{m\in M}([b] \sqcup [m]) = \bigsqcap_{m\in M}[b \sqcup m] = [b \sqcup \bigsqcap M] = [b] \sqcup [\bigsqcap M] = [b] \sqcup \bigsqcap[M]$, that is, $[M]$ is $\bigsqcap$-dis in $A/(C,D)$. It is verified similarly that $\bigsqcup[N] = [\bigsqcup N]$ and $[N]$ is $\bigsqcup$-dis in $A/(C,D)$ for any $N \in R$.

(b) $\Rightarrow$ (a). Suppose $a, b \in A$ and $M \in Q$. If $\nabla_{C\cup\{a\}} \cap \Delta_{D\cup\{b,m\}} \neq \emptyset$ for any $m \in M$, then $[a] \sqsubseteq [b \sqcup m](= [b] \sqcup [m])$ for any $m \in M$. By (b),$[a] \sqsubseteq \bigsqcap_{m\in M}([b] \sqcup [m]) = [b] \sqcup \bigsqcap[M] = [b] \sqcup [\bigsqcap M] = [b \sqcup \bigsqcap M]$. Thus, $a \leqslant b \sqcup \bigsqcap M \ mod(C,D)$, namely, $\nabla_{C\cup\{a\}} \cap \Delta_{D\cup\{b,\sqcap M\}} \neq \emptyset$. The case for $N \in R$ is similar.

COROLLARY. *Suppose A is a (Q,R) complete distributive lattice. Then the following are equivalent.*

(a) *(Q,R) is distributive.*

(b) *For any $a, b \in A$, $(\{a\},\{b\})$ is (Q,R) complete.*

PROOF. Take $(\emptyset,\emptyset)$ for (C,D) in the lemma.

Suppose A is distributive lattice and let $\mathcal{H}$ be the set of all prime filters in A. (For the sake of convenience, let $\mathcal{H} = \{A\}$ if A is a singleton).)

It is well known that

$$\begin{array}{cccl} * : & A & \longrightarrow & 2^{\mathcal{H}} \\ & \in & & \in \\ & a & \longrightarrow & a^* = \{F : a \in F \text{ and } F \in \mathcal{H}\} \end{array}$$

is an embedding. Let $B(A)$ be the Boolean algebra generated by A in $2^{\mathcal{H}}$. In [1], Görnemann proved

Lemma 2. *Suppose A is a (Q, R) complete distributive lattice. Then the following are equivalent:*

(a) *(Q, R) is distributive.*
(b) *$\forall M \in Q(\bigsqcap M)^* = \bigsqcap M^*$ and $\forall N \in R(\bigsqcup N)^* = \bigsqcup N^*$ in $B(A)$.*

Remark. This is proved in [1] for bounded distributive lattice but the condition that A is bounded is optional.

Theorem. *Suppose that A is a (Q, R) complete distributive lattice, $|Q \cup R| \leqslant \omega$, and $C, D \subseteq A$. Then the following are equivalent:*

(a) *For any $a, b \in A$, $(C \cup \{a\}, D \cup \{b\})$ is (Q, R) complete.*
(b) *For any $a, b \in A$, $a \leqslant b mod(C, D)$ or there is a prime filter F in A such that $(F, A - F)$ is (Q, R) complete, $C \cup \{a\} \subseteq F$, and $D \cup \{b\} \subseteq A - F$.*

Remark. A filter F is called R saturated if $\forall N \in R \bigsqcup N \in F \Rightarrow N \cap F \neq \emptyset$. If F is a prime filter, then F is Q complete R saturated iff $(F, A - F)$ is (Q, R) complete.

Proof. (a) $\Rightarrow$ (b). Suppose not $a \leqslant bmod(C, D)$. Then $[a] \not\sqsubseteq [b]$ in $A/(C, D)$ and $[a]^* \not\sqsubseteq [b]^*$ in $B/(C, D))$. By the above lemmas, $B(A/(C, D))$ is a $([Q]^*, [R]^*)$ complete Boolean algebra and, of course, $|[Q]^* \cup [R]^*| \leqslant \omega$. Since $[a]^* \not\subseteq [b]^*$ in $B(A/(C, D))$, by the Rasiowa-Sikorski Lemma for Boolean algebras, there is a $[Q]^*$ complete $[R]^*$ saturated ultrafilter U in $B(A/(C, D))$ such that $[a]^* \in U$ and $[b]^* \notin U$. Let $F = \{x \in A : [x]^* \in U\}$. It is trivial that $a \in F$ and $b \notin F$. If $x \in C$, then $[x]^* = \mathbf{1}$ in $B(A/(C, D))$ and $[x]^* \in U$. So $C \subseteq F$. Similarly we have $D \cap F = \emptyset$. Suppose $x, y \in F$. Then $[x]^*$, $[y]^* \in U$. Since $[x \sqcap y]^* = ([x] \sqcap [y])^* = [x]^* \sqcap [y]^* \in U$, we have that $x \sqcap y \in F$. In a similar way it is shown that F is a prime filter. Now, suppose $M \in Q$ and $\nabla_F \cap \Delta_{(A-F)\cup\{m\}} = \emptyset$ for any $m \in M$. Then $m \in M$. Then $m \in F$ for any $m \in M$, since $\nabla_F = F$ and $\Delta_{A-F} = A - F$. Thus we have that $[m]^* \in U$ for any $m \in M$ by the definition of F. But U is $[Q]^*$ complete. So $\bigsqcap [M]^* \in U$, form which it follows that $[\bigsqcap M]^* \in U$, since $\bigsqcap [M]^* = (\bigsqcap [M]^) * = [\bigsqcap M]^*$ by the lemmas. Hence $\bigsqcap M \in F$, namely, $\nabla_F \cap \Delta_{(A-F)\cup\{\bigsqcap M\}} = \emptyset$. The case for $N \in R$ is similar.

(b) $\Rightarrow$ (a). Suppose $a, b \in A$ and $M \in Q$.
If $\nabla_{C\cup\{a\}} \cap \Delta_{D\cup\{b,\bigsqcap M\}} = \emptyset$, then, since not $a \leqslant b \sqcup \bigsqcap M$ $mod(C, D)$, there is some prime filter F in A such that $(F, A - F)$ is (Q, R) complete, $C \cup \{a\} \subseteq F$,

and $D \cup \{b \sqcup \bigsqcap M\} \subseteq A - F$. Clearly $\nabla_F = F$ and $\Delta_{(A-F)\cup\{\bigsqcap M\}} = A - F$. So $\nabla_F \cap \Delta_{(A-F)\cup\{\bigsqcap M\}} = \emptyset$ and we have that $\nabla_F \cap \Delta_{(A-F)\cup\{m\}} = \emptyset$ for some $m \in M$, since $(F, A - F)$ is (Q, R) complete. It is trivial that $\nabla_{C\cup\{a\}} \cap \Delta_{D\cup\{b,m\}} = \emptyset$ since $C \cup \{a\} \subseteq F$ and $D \cup \{b, m\} \subseteq (A - F) \cup \{m\}$ $(= A - F)$. The case for $N \in R$ is verified similarly.

Corollary. *Suppose* A *is a* (Q, R) *complete distributive lattice and* $|Q \cup R| \leqslant \omega$. *Then the following are equivalent:*

(a) (Q, R) *is distributive.*

(b) *For any* $a, b \in A$, $a \subseteq b$ *or there is a prime filter* F *in* A *such that* $(F, A - F)$ *is* (Q, R) *complete,* $a \subseteq F$, *and* $b \notin F$.

Proof. Take $(\emptyset, \emptyset)$ for (C, D) in the theorem and apply the Corollary of Lemma 1.

References

[1] S. Görnemann, *A logic stronger than intuitionism*, **The Journal of Symbolic Logic** 36 (1971).

[2] C. Rauszer and B. Sabalski, *Notes on Rasiowa-Sikorski Lemma*, **Studia Logica** 34 (1975).

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