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SOME FORMAL PROPERTIES OF OBJECTIVES

In [1] the counterpart of entailment, and the objectives of conjunction and disjunction have determined as follows:

[T1] $\alpha \Rightarrow \beta \equiv S(\alpha)$ involves $S(\beta)$
[T2] $S(\alpha \wedge \beta) = Min/S(\alpha) \cdot S(\beta/$
[T3] $S(\alpha \vee \beta) = Min/S(\alpha) \cup S(\beta)/.$

Let us see now that the operations in T2 and T3 have indeed all the requisite formal properties, i.e. that under them the objectives of propositions form a distributive lattice. (They will be written for short "$S(\alpha) * S(\beta)$" and "$S(\alpha) + S(\beta)$" respectively).

We consider first some lemmas on the *minimum* $Min(A)$ as defined in [1]. It follows immediately from the definition that

[M1] $A \subset B \rightarrow A \cap Min(B) \subset Min(A)$

for any $A, B \subset SE$. Obvious corollaries of M1 are

[M2] $A \cap (A + B) \subset Min(A)$
[M3] $(A + B) \subset Min(A) \cup Min(B)$.

From M1 and M3 it follows by straight set algebra that

[M4] $(A + B) \subset (A + Min(B))$,

and it turns out that the converse of M4 holds too:

[M5] $(A + Min(B)) \subset (A + B)$.

For assume $x \in (A + Min(B))$, and suppose $x \notin (A + B)$. By assumption $x \in A \cup B$, and neither A, nor $Min(B)$ contains any x' smaller than x.

By assumption, however, there should be some x' such that $x' < x$ and $x' \in (A + B)$, which is impossible. Thus in view of M4 and M5

[M6] $(A + Min(B)) = (A + B)$

and consequently

[M7] $(Min(A) + Min(B)) = (A + B)$.

From M7 the equality T3 follows easily by the axiom A5. To see

[M8] B involves $A \to (A + B) = Min(A)$

assume the antecedent, i.e. that for any $y \in B$ there is some $x \in A$ such that $x \leqslant y$. Now suppose first that $x \in Min(A)$. Thus $x \in A \cup B$. If, however, $x \notin (A + B)$, then there should be some x' such that $x' \in B \wedge x' < x$. But then by assumption there should be also some $x'' < x$, which would contradict the supposition. Thus $Min(A) \subset (A + B)$. Conversely, suppose $x \in (A + B)$. Thus $x \in A \vee x \in B$. If $x \in A$, then $x \in Min(A)$ by M2. If $x \in B$, then by assumption there is some x' such that $x' \in A$ and $x' \leqslant x$, i.e. such that $(x' \in A \wedge x' < x) \vee (x' \in A \wedge x' = x)$. Now if the latter, then $x \in A$, and thus $x \in Min(A)$ by M2. If the former, then $x \in (A + B) \wedge x' < x \wedge x' \in A$. But if $x' \in A$, then $x' \in A \cup B$, and since $x' < x$, so $x \notin (A + B)$, which contradicts the supposition. Thus $(A + B) \subset Min(A)$.

Since – for any $A, B \subset SE$ – $A \cdot B$ involves A, we get as a corollary of M8:

[M9] $(A + A \cdot B) = Min(A)$.

Consider next three lemmas concerned with involvement. From the definition of the product $A \cdot B$, and in view of the partial ordering of SE, we have

[In1] A involves $B \longrightarrow A \cdot C$ involves $B \cdot C$.

Extending the axiom A7 to cover arbitrary SE-sets, we get

[In2] B involves $Min(B)$.

And the immediate consequence of In1 and In2 is

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[In3] $A \cdot B$ involves $A \cdot Min(B)$.

Now substituting in M8 $A \cdot B/B$ and $A \cdot Min(B)/A$ we get by In3: $(A \cdot B + A \cdot Min(B)) = (A * Min(B))$. Since, however, $Min(B) \subset B$, and since the product is distributive over the union (i.e., $A \cdot /B \cup C? = A \cdot B \cup C$; cf. [2], chapter 2, where it is called "the product of subsets"), we also have: $(A * B) = (A * /B \cup Min(B)/) = (A \cdot B + A \cdot Min(B))$. Thus we get

[M10] $(A * B) = (A * Min(B))$

and consequently

[M11] $(A * B) = (Min(A) * Min(B))$

which by A4 and the theorem $V(\alpha) \cdot v(\beta) = V(\alpha) \cap V(\beta)$ yields immediately the equality T2.

Finally from M11, and from M9 by the inclusion $A \subset A \cdot A$, we get

[M12] $(Min(A) * Min(A)) = (A * A) = Min(A)$,

and from M7 we have

[M13] $(Min(A) + Min(A)) = (A + A) = Min(A)$.

Now at last we are in a position to show the main points. Let us call the operations $(A * B) =_{Df} Min(A \cdot B)$, $(A + B) =_{Df} Min(A \cup B)$ used so far the *quasi-product* and the *quasi-sum* respectively. Both are evidently commutative. In view of M6 the quasi-sum is associative, and so is the quasiproduct, in view of M10, and the product being associative (cf. [2]).

If the domain is restricted to the family of *Minimal sets*, i.e. to such $A \subset SE$ that $A = Min(A)$, then both operations are also idempotent by M12 and M13; i.e.,

$$(A * A) = A \equiv A = Min(A)$$
$$(A + A) = A \equiv A = Min(A).$$

In our case this restriction makes no difference, the family of objectives being by definition included in that of minimal sets.

Both operations are distributive with respect to each other. The equality

[D1] $(A * (B + C) = ((A * B) + (A * C))$

follows from M10 and M7, for any $A, B, C \subset SE$. To see, on the other hand, the equality

[D2] $(A + (B * C)) = ((A + B) * (A + C))$

consider, starting from the right, the following steps:

$$
\begin{aligned}
&((A + B) * (A + C)) \\
&\quad = ((A * A)+(A * B)+(A * C) + (B * C)) && \text{[by D1]} \\
&\quad = (Min(A)+(A * B)+(A * C)+(B * C)) && \text{[by M12]} \\
&\quad = (Min(A) + (A * B) + Min(A) + (A * C) + (B * C)) && \text{[by M13]} \\
&\quad = (A + A \cdot B + A + A \cdot C + B \cdot C) && \text{[by M7]} \\
&\quad = (Min(A) + Min(A) + B \cdot C) && \text{[by M9]} \\
&\quad = (Min(A) + B \cdot C) && \text{[by M13]} \\
&\quad = (A + (B * C)) && \text{[by twice M6].}
\end{aligned}
$$

The absorption laws $(A + (A * B)) = A = (A * (A + B))$ hold for any $B \subset SE$, but again provided that A is a minimal set. For we have $(A+(A*B)) = (A+A\cdot B) = Min(A)$, by M6 and M9; and $(A*(A+B)) = ((A * A) + (A * B)) = (Min(A) + (A * B)) = (A + A \cdot B) = Min(A)$, by S1, M12, M7, and M9.

Denoting the family of objectives by "OB", and that of minimal sets by "Min", we have $OB \subset Min$. Thus the structure $\langle OB, +, * \rangle$ is a distributive lattice, but it is not a Boolean algebra, for there is no complementation. How to provide for it, i.e. how to determine the objective of negation, will be discussed in another paper.

References

[1] B. Wolniewicz, *Objectives of Propositions*, this **Bulletin**, vol. 7, no. 3 (1978), pp. 143–147.

[2] E. S. Lyapin, **Semigroups** (in Russian), Moskva 1960.

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