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MORE ABOUT THE LATTICE OF TENSE LOGIC

Let Ł denote the propositional language in $\neg, \wedge, \vee, \rightarrow, \Box\!\!-, -\!\!\Box$ where $\Box\!\!-, -\!\!\Box$ are unary. $\Box\!\!- P$ is read: "it will always be the case..." and $-\!\!\Box P$: "it has always been the case...". O abbreviates $\wedge\neg p$, where p is a fixed variable. NZ denotes the lattice of all normal tense logics in Ł whose smallest logic is denoted by Kt (cf. e.g. [6]). NZ is a distributive algebraic lattice, cf. [3]. For $L^0 \in NZ$ put $EL := \{L \in NZ | L \supseteq L^0\}$. $L \in NZ$ has *dimesion* n, $dim\ L = n$ $(n \in \omega)$, if there is a chain $L = L_0 \subset L_1 \subset \ldots \subset L_n =$ Ł $(L_i \in NZ)$, which cannot be refined. Since NZ is modular, $dimL$ is uniquely determined if it exists. The $L \in NZ$ with $dimL = 1$ coincide with the POST-complete $L \in NZ$. There are non-tabular POST-complete $L\ NZ$, as follows from a result in [6]. Below we characterize all POST-complete tabular $L \in NZ$. Furthemore we describe all splittings of NZ and then add some remarks on $ES4t$, where $S4t$ is the tense version of $S4$. $L \in NZ$ is called *prime* or meet-irreducible if $L = L_1 \cap L_2$ implies $L = L_1$ or $L = L_2$, and *totally prime* if $L \bigcap_{i \in L} L_i$ implies $L = L_i$ for some $i \in I$, where I is any index set and $L_i \in NZ$. If EL satisfies the upper chain condition (this is the case e.g. for all tabular and pretabular $L \in NZ$), then L has a unique shortest prime decomposition.

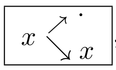
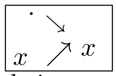
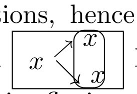
NZ corresponds to the lattice of subvarieties of the variety ZBA of *tense algebras*, defined to be Boolean algebras A with two additional operators $\blacksquare\!\!-$ and $-\!\!\blacksquare$, such that $(A, \blacksquare\!\!-)$ and $(A, -\!\!\blacksquare)$ are both normal modal algebras and in addition $\blacksquare\!\!-$, $-\!\!\blacksquare$ are *conjugated*, i.e.
$\blacksquare\!\!- aUb = 1 \leftrightarrow aU -\!\!\blacksquare b = 1$ $(a, b \in A)$ (cf. [2] or [6]). $LA \in NZ$ denotes the tense logic determined by A, A being a matrix with designated element 1. $MdL := \{A \in ZBA | L \subseteq LA\}$. For $\mathbb{K} \subseteq ZBA$ let $\mathbb{K}_{s.i.}$ denote the class of subdirect irreducible members of $\mathbb{K}$, $\mathbb{K}_{f.s.i.}$ the class of finite $A \in \mathbb{K}_{s.i.}$, and $\mathbb{K}_{d.i.}$ the class of direct irreducible members of $\mathbb{K}$. $A \in ZBA$ is *H-simple* if

A is non-trivial and has no non-trivial homomorphism; and S-*simple* if it has no non-trivial subalgebras. Let G denote the class of all structures g with one binary relation $\lhd$, and Gf the finite $g \in G$. To each $g \in G$ there corresponds its tense algebra Ag, defined to be the algebra of subsets of g, together with the operations $\blacksquare\!\!-$, $-\!\!\blacksquare$ such that for $a \subseteq g \blacksquare\!\!- \, a = \{S \in g | S \lhd S' \Rightarrow S' \in a$ for all $S' \in g\}$ and $-\!\!\blacksquare a = \{S \in g | S' \lhd S \Rightarrow S' \in a$ for all $S' \in g$. Put $Lg = LAg$. $MsL := \{g \in G | L \subseteq Lg\}$ denotes the class of L-*frames* or *model structure* for L.

PROPOSITION 1. *For finite $A \in ZBA$ the following conditions are equivalent: (i) $A \in ZBA_{s.i.}$, (ii) $A \in ZBA_{d.i.}$, (iii) A is H-simple, (iv) $A \simeq Ag$ for some connected $g \in Gf$, (v) LA) is prime, (vi) LA is totally prime.*

The proof is partly based on JÒNSSON's theory of congruence distributive varieties. Let NZ^τ denote the sublattice of tabular logics in NZ. All $L \in NZ^\tau$ have finite dimension, since EL is finite in this case (cf. [5]).

PROPOSITION 2. *For $L \in NZ^\tau$ the following are equivalent: (i) L is POST-complete, (ii) $L = LA$ for some S-simple $A \in ZBA_{f.d.i.}$, (iii) $L = Lg$ for some connected $g \in Gf$ permitting no contraction. Here a contraction of g is a non-trivial partition $(k_S)_{S \in g}$ of g, such that if $S \lhd T$ then to each $S' \in k_S$ there is some $T' \in k_T$ with $S' \lhd T'$ and for each $T' \in k_T$ there is some $S' \in k_S$ such that $S' \lhd T'$.*

EXAMPLES. $\boxed{\cdot \to x}$, $\boxed{x \to x}$, ,  and infinitely many other graphs have no contractions, hence their tense logics are POST-complete in NZ. $\boxed{(x \to \cdot)}$ and  have contractions as indicated. Here $\cdot$ denotes a reflexive, x an irreflexive node.

For $P \in$ Ł let $\boxed{n}\,P$ denote the conjunction of all formulas $\Box_1 \ldots \Box_n P$, where each $\Box_i$ is either $\Box\!\!-$ or $-\!\!\Box$ $\boxed{0}\,P = P$. Put $L^n := Kt(\boxed{n}\,0)$.

PROPOSITION 3. $Kt = \bigcap_{n \in \omega} L^n$.

The proof is not easy and uses techniques of "frame ramification", which are discussed in detail in [5].

$A \in MdL^0$ ($L^0 \in NZ$) is said to *split* EL^0 if there is some $L^1 \in EL^0$ such that EL^0 is the disjoint union of $\{L \in EL^0 | L \subseteq LA\}$ and EL^1. If

L^0 has the finite model property (as e.g. Kt), then A splits EL^0 only if $A \in Md_{f.s.i.}L^0$. The next theorem generalizes a beautiful result of [1].

PROPOSITION 4. *$A \in ZBA$ splits $NZ = EKt$ iff $A \in ZBA_{f.s.i.}$ and $\boxed{n}\, 0 \in A$ for some $n \in \omega$.*

EXAMPLE $\boxed{1}\, 0 \in L\, \boxed{x}$. Hence $NF/\boxed{x}$ is well defined and turns out to be the tense logic $F := Kt(\Diamond\!\!-p \vee -\!\!\Diamond p \vee \Diamond\!\!- \neg p \vee -\!\!\Diamond \neg p)$, which turns out to be complete with respect to the class $Gfe \backslash \{\, \boxed{x}\, \}$, where Gfe denotes the class of all finite connected graphs. The modal reduct of F is K, and not the denotic logic $D = N/\boxed{x}$, as one might except.

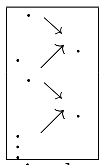
REMARK. If $\boxed{m}\, 0 \in LA$, $A \in ZBA$ finite, then A has a representation $A = Ag$, where g does not contain a circle and each path has length $\leqslant n$.

Put $ⓜ P = P \wedge \boxed{1}\, P \wedge \ldots \wedge \boxed{m}\, P$ and $Kt^m := Kt(ⓜp \to \Box\!\!- \boxed{m}\, p \wedge -\!\!\Box\, \boxed{m}\, p)$. Frames for Kt^m are those $g \in G$ for which if $S, T \in g$ are connected, then they are connected already by a path of length $\leqslant m$ (no matter what direction the edges have).

PROPOSITION 5. *If $L^0 \supseteq Kt^m$ for some m, then each $A \in Md_{f.s.i.}L^0$ splits EL^0.*

There is a corresponding result of the another for N (cf. [4]), but the proof is different. The splitting logic L^0/A is $L^0(\curlywedge A)$, where $\curlywedge A := \boxed{m}\, \delta A \to p_0$. Here δA is the Jankov diagram of A (cf. [4]), expressing the structure of A in a formula. p_0 is the variable corresponding to the zero of A.

Since e.g. $S4t$, $K4t \supseteq Kt^1$, each $A \in Md_{f.s.i.}S4t$ splits $ES4t$. The same holds for $K4t$ and the tense version of many other modal logics. In particular, this applies to numerous tense logics with transitive linear time.

Each non tabular $L \in NZ$ is contained in some pretabular $L \in NZ$ as in the case of N. N has infinitely many pretabular logics, but $S4$ has only five, as is well known. It may therefore be asked how many pretabular logics $ES4t$ contains. The answer is "infinitely many". E.g. L  is pretabular and of dimension 3, where in $ES4$ all pretabular logics lay "behind" the tabular logics and hence have no finite dimension.

By the way, it should be observed that $L\boxed{\cdot}$ is the only POST-complete logic in $ES4t$, like in the case $S4$. But in contrast to the picture of the upper part of $ES4$ (cf. [4]), the tense logic $L\boxed{\cdot \to \cdot}$ has already infinite many tabular *ip*'s (= immediate predecessors) in $ES4t$, for there are infinitely many finite connected orders, admitting only contractions to $\boxed{\cdot \to \cdot}$ and $\boxed{\cdot}$. In $ES4$ each tabular logic has finitely many *ip*'s only, as is well known.

It can be shown that in $ES4t$ the logics of $\boxed{\cdot \to \cdot}$ and $\boxed{\cdot \overset{\curvearrowright}{\leftarrow} \cdot}$ are the only tabular *ip*'s of $L\boxed{\cdot}$. Moreover, these two logics are the only tabular *ip*'s of $L\boxed{\cdot}$. It would be, in a way, very surprising, if $L\boxed{\cdot}$ has an incomplete *ip* in $ES4t$. It can easily be proved that $S4/\boxed{\cdot \to \cdot} = S5$ and $S4/\boxed{\cdot \overset{\curvearrowright}{\leftarrow} \cdot} = S4.\ 1 (= S4(\Box\Diamond p \to \Diamond\Box p))$ in the case of modal logic. But we were unable to decide whether $S4t/\boxed{\cdot \to \cdot}$ and $S4t/\boxed{\cdot \overset{\curvearrowright}{\leftarrow} \cdot}$ are complete or not. If these two logics were complete, then $L\boxed{\cdot \to \cdot}$ would have in fact the only *ip*'s $L\boxed{\cdot \to \cdot}$ and $\boxed{\cdot \overset{\curvearrowright}{\leftarrow} \cdot}$ in $ES4t$. Let us finally mention the problem, whether the pretabular $L \in ES4t$ are complete with respect to Kripke-semantics. This would be the case if $S4t/a_n$ was locally finite, where a_n denotes the linear order of $n+1$ points. Up to now, no really rich locally finite varieties of $S4t$-algebras are known. Clearly, if MdL ($L \in NZ$) is locally finite, then L is essentially complete, i.e. each $L' \in EL$ is complete. Maximova has shown (Algebra i Logika 14 (1975), pp. 304–319, Theorem 2) that $L \in ES4$ is locally finite iff $L \in ES4^n$ for some n, where $S4^n$ is the basic logic of the nth layer. Since $S4^n = S4/a_n$, this means that $L \in ES4$ is locally finite *iff* L omits one of the frames a_n. Is this true also for $S4t$? Another question is, whether each $L \in ES4t$ of finite dimension is finitely axiomatizable. This is clearly the case for $ES4$, for then the tabular logics are just the logics of finite dimension.

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