Bulletin of the **S**ection of **L**ogic
Volume 8/1 (1979), pp. 27–29

reedition 2010 [original edition, pp. 27–29]

Martin Tabakov

A FORMAL SYSTEM WITHOUT WELL-FORMED FORMULAS

This is an abstract of the paper submitted to "Problems of Logik". V.VII. Sofia.

The notion of a well-formed formula (wff) is not central to a formal axiomatic system (FAS), because one constructs a system with the view to specify and isolate the theorems. We shall present a formal system T of propositional calculus that works without the concept of a wff.

§1. We first adopt a FAS L of propositional calculus which will be proved to be equivalent to our new system T.

The system L

a) Vocabulary of L: $a_1, a_2, a_3, \ldots, (,), \neg, \rightarrow$

b) Formation rules:

  b.1) a_i is a wff

  b.2) if p is wff so is $(\neg p)$

  b.3) if p and q are wffs so is $(p \rightarrow q)$

  b.4) p is a wff if and only if it can be constructed by a finite number of applications of the above rules.

c) Axioms: If p, q, r, are wffs, the following wffs are axioms:
$A1(p,q) \leqslant (p \rightarrow (q \rightarrow p))$
$A2(p,q,r) \leqslant ((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$
$A3(p) \leqslant ((\neg(\neg p)) \rightarrow p)$
$A4(p) \leqslant (p \rightarrow (\neg(\neg p)))$
$A5(p,q) \leqslant ((p \rightarrow q) \rightarrow ((\neg p) \rightarrow (\neg p)))$

d) Rules of inference – Modus ponens

It is easily proved that L is equivalent to system P in [1], II, §27.

We propose a system T which is constructed as follows:

a) The primitive symbols of the vocabulary are identical with the primitive symbols of L.
Any finite sequence of symbols is called a string.
Strings will be designated by: $x, y, z \dots$
A short notation for $(x \to x)$ will by $\overline{x}$ for any string x.

b) Axioms: the string $\overline{a}_i$ is an axiom for every i

c) Rules of inference:
P1. $x, (x \to y) \vdash y$
P2. $\overline{x}, \overline{y} \vdash A1(x, y)$
P3. $\overline{x}, \overline{y}, \overline{z} \vdash A2(x, y, z)$
P4. $\overline{x} \vdash A4(x)$
P5. $\overline{x} \vdash A3(x)$
P6. $\overline{x}, \overline{y} \vdash A5(x, y)$
where $A1, A2, A3, A4, A5$ are axioms of L according to 1.c).

§2. We will prove that T is equivalent to L

LEMMA 1. *If p and q are $wffs$ and x is a string in which the number of left parentheses is equal to the number of right parentheses and if $(p \to q) \lesseqgtr (x \to y)$, then plessgtrx; $q \lesseqgtr y$.*

LEMMA 2. *If $(p \to p)$ is a wff, then p is a wff.*

LEMMA 3. *If $(p \to q)$ is a wff and p is a wff, then q is a wff.*

LEMMA 4. *If x is a string of T, then $T, \overline{x} \vdash \overline{(\neg x)}$.*

LEMMA 5. *If x and y are strings of T, then $T, \overline{x}, \overline{y} \vdash \overline{(x \to y)}$.*

LEMMA 6. *If p is a wff, then $T \vdash \overline{p}$.*

It is proved by induction. Here we use Lemma 4 and Lemma 5.

LEMMA 7. *If $L \vdash p$, then $T \vdash p$.*

It is proved by induction using Lemma 6.

LEMMA 8. *If $T \vdash x$, then $L \vdash x$.*

Here we use Lemmas 2 and 3.
From Lemma 7 and Lemma 8 we obtain

THEOREM: *$L \vdash p$ iff, when $T \vdash p$.*

It should be pointed out that the vocabulary of system T contains no addidtional symbols, i.e. it is identical with the vocabulary of L.

References

[1] A. Church, **Introduction to mathematical logic**, Princeton, 1956.

Section of Logic
Institute of Philosophy
Bulgarian Academy of Sciences