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Małgorzata Porębska

ON IMPLIVALENTIAL ALGEBRAS

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Let E denote the equational class of equational algebras (see [4]) and I – the equational class of implicational algebras (see [1]). Our task is to give a characterization of the smallest variety EI containing E and I. By implivalential algebras we shall mean algebras of the variety EI.

We use the common algebraic notation. In particular the German capitals: $\mathcal{A}, \mathcal{D}, \mathcal{L}, \dots$ will be used for algebras, and the corresponding Latin capitals: $A, B, C, \dots$ for their domains. Writing algebraic equations we adopt the convention of associating to the left and ignoring the sign of binary operation: for example we will write $a_1 a_2 \dots a_n$ instead of $(\dots(a_1 * a_2)\dots) * a_n$.

Let us recall [3] that equational classes K_1, K_2 of algebras of the same type are independent if there exists a binary polynomial symbol $p(x, y)$ such that $p(x, y) = x \in Id(K_1)$ and $p(x, y) = y \in Id(K_2)$.

Let us observe that $x(yy)(y(xx)y) = x \in Id(E)$ and $x(yy)(y(xx)y) = y \in Id(I)$. This means that the following theorem holds:

Theorem 1. *E and I are independent.*

The direct product $E \times I$ is the class of all algebras $\mathcal{A}$ which are isomorphic to an algebra of the form $\mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_1 \in E$, $\mathcal{A}_2 \in I$. From the above theorem and Theorem 1 [3] we conclude that

COROLLARY 1. $EI = E \times I$.

The following theorem gives an axiomatization for EI:

THEOREM 2. *$\mathcal{A} \in EI$ if and only if for arbitrary $a, b, c \in A$ the following equations hold:*

(ei 1) $aab = b$,
(ei 2) $a(bb)(a(bb)a) = a$,
(ei 3) $ab(cc)(ab) = a(cc)a(b(cc)b)$,
(ei 4) $a(b(ca)) = a(c(ba))$,
(ei 5) $a(a(bc)) = ab(ac)$,
(ei 6) $abb(abb(ac)) = ac$,
(ei 7) $ab(ab(baa)) = ba(abb)$.

PROOF. Assume that algebra $\mathcal{A}$ satisfies (ei 1) – (ei 7). Let us define two relations on $A \times A$:

$$\theta_E = \{\langle a, b\rangle : a(bb)(b(aa)b) = b\}$$
$$\theta_I = \{\langle a, b\rangle : a(bb)(b(aa)b) = a\}$$

which are congruence of the algebra $\mathcal{A}$ such that:

1^0 $\theta_E \cap \theta_I = \omega_{\mathcal{A}}$, where $\omega_{\mathcal{A}}$ is the smallest congruence of $\mathcal{A}$,
2^0 $\theta_E \vee \theta_I = \iota_{\mathcal{A}}$, where $\iota_{\mathcal{A}}$ is the smallest congruence of $\mathcal{A}$,
3^0 $\theta_E\theta_I = \theta_I\theta_E$, what means that θ_E and θ_I are permutable,
4^0 $\mathcal{A}/\theta_E \in E, \mathcal{A}/\theta_I \in I$ and θ_E, θ_I are the smallest congruence relations having this property.

Now from Theorem 19.3 [2] we inference that $\mathcal{A} \cong \mathcal{A}/\theta_E \times \mathcal{A}/\theta_I$, so $\mathcal{A} \in EI$. Simple computations showing that (ei 1) – (ei 7) hold in all $E-$ and $I-$algebras complete the proof of this theorem.

The following arbitrary EI-algebra:

THEOREM 3. *If $\mathcal{A}_1 \in E$, $\mathcal{A}_2 \in I$, then $\mathcal{L}(\mathcal{A}_1 \times \mathcal{A}_2) \cong \mathcal{L}(\mathcal{A}_1) \times \mathcal{L}(\mathcal{A}_2)$.*

PROOF. Assume that $\mathcal{D} \in EI$. For any congruence φ of algebra $\mathcal{D}$ we define two relations on $B \times B$:

$$\varphi_1 = \{\langle a, b\rangle : \langle a(bb)(b(aa)b), b\rangle \in \varphi\}$$
$$\varphi_2 = \{\langle a, b\rangle : \langle a(bb)(b(aa)b), a\rangle \in \varphi\}$$

which are congruence of $\mathcal{D}$ such that:

1^0 $\varphi_1 \wedge \varphi_2 = \varphi$,
2^0 $\varphi_{12} = \varphi_{21} = \iota_{\mathcal{D}}$,
3^0 $\varphi_{11} = \varphi_1, \varphi_{22} = \varphi_2$,
4^0 $(\varphi \cap \psi)_1 = \varphi_1 \cap \psi_1, (\varphi \cap \psi)_2 = \varphi_2 \cap \psi_2$,
5^0 $(\varphi \vee \psi)_1 = \varphi_1 \vee \psi_1, (\varphi \vee \psi)_2 = \varphi_2 \vee \psi_2$.

Let $\mathcal{L}(\mathcal{D})$ denote congruence lattice of algebra $\mathcal{D}$. Further we define two relations on $C(\mathcal{D}) \times C(\mathcal{D})$:

$$\Delta' = \{\langle \varphi, \psi \rangle : \varphi_1 = \psi_1\}$$

$$\Delta'' = \{\langle \varphi, \psi \rangle : \varphi_2 = \psi_2\}$$

which are congruences of $\mathcal{L}(\mathcal{D})$ and fulfill the following conditions:

1^0 $\Delta' \cap \Delta'' = \omega_{\mathcal{L}(\mathcal{D})}$,
2^0 $\Delta' \vee \Delta'' = \iota_{\mathcal{L}(\mathcal{D})}$,
3^0 $\Delta'\Delta'' = \Delta''\Delta'$.

From Theorem 19.3 [2] we know that $(\mathcal{D}) \cong \mathcal{L}(\mathcal{D})/\Delta' \times \mathcal{L}(\mathcal{D})/\Delta''$. Relations θ_E, θ_I (defined on $\mathcal{D}$ in the same way as in the proof of the previous theorem) are elements of the set $C(\mathcal{D})$, so let us define two subsets of $C(\mathcal{D})$:

$$[\theta_E) = \{\varphi : \varphi \in C(\mathcal{D}), \theta_E \subseteq \varphi\}$$

$$[\theta_I) = \{\varphi : \varphi \in C(\mathcal{D}), \theta_I \subseteq \varphi\}$$

$\langle [\theta_E), \subseteq \rangle$ and $\langle [\theta_I), \subseteq \rangle$ are sublattices of the lattice $\mathcal{L}(\mathcal{D})$. It is easy to check that:

$$\langle [\theta_E, \subseteq \rangle \cong \mathcal{L}(\mathcal{D})/\Delta' \ ; \ \langle [\theta_I, \subseteq \rangle \cong \mathcal{L}(\mathcal{D})/\Delta''$$

We set up a required isomorphisms h_1, h_2 as follows:

$$h_1([\varphi]\Delta') = \varphi_1 \ ; \ h_2([\varphi]\Delta'') = \varphi_2.$$

Now from Theorem 11.3 [2] we conclude that for an arbitrary algebra $\mathcal{D} \in EI : \mathcal{L}(\mathcal{D}) \cong \mathcal{L}(\mathcal{D}/\theta_E) \times \mathcal{L}(\mathcal{L}/\theta_I)$. If we assume that $\mathcal{A}_1 \in E$, $\mathcal{A}_2 \in I$, then $\mathcal{A}_1 \times \mathcal{A}_2 \in EI$. So $\mathcal{L}(\mathcal{A}_1 \times \mathcal{A}_2) = \mathcal{L}((\mathcal{A}_1 \times \mathcal{A}_2)/\theta_E) \times \mathcal{L}((\mathcal{A}_1 \times \mathcal{A}_2)/\theta_I)$, where θ_E, θ_I are defined on $\mathcal{A}_1 \times \mathcal{A}_2$ in the same way as in the proof of Theorem 2. Because of the minimal property θ_E, θ_I: $(\mathcal{A}_1 \times \mathcal{A}_2)/\theta_E = \mathcal{A}_1$ and $(\mathcal{A}_1 \times \mathcal{A}_2)/\theta_I = \mathcal{A}_2$, so we conclude that $\mathcal{L}(\mathcal{A}_1 \times \mathcal{A}_2) \cong \mathcal{L}(\mathcal{A}_1) \times \mathcal{L}(\mathcal{A}_2)$.

Since all E- and I-algebras have a modular congruence lattices, from above theorem and from Theorem 1 [3] we immediately have the following two corollaries:

Corollary 2. *Each EI-algebra have a modul congruence lattice.*

Corollary 3. *An arbitrary algebra $\mathcal{A} \in EI$ has, up to isomorphism, a unique representation in $E \times I : \mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_1 \in E$, $\mathcal{A}_2 \in I$.*

Let $\mathcal{F}^n_{EI}$ denote a free algebra free-generated by an n-elementary set over some class K. We can prove:

Theorem 4. $\mathcal{F}^n_{EI} \cong \mathcal{F}^n_E \times \mathcal{F}^n_I$.

Proof. Assume that $\mathcal{F}^n_{EI}$ is a free algebra over EI, free-generated by an n-elementary set $\{a_1, \ldots, a_n\}$. Observe that θ_E, θ_I, defined on $\mathcal{F}^n_{EI}$ in the same way as in the proof of Theorem 2, are fully invariant congruence relations. So from Theorem 25.6 [2] we have that $\mathcal{F}^n_{EI}/\theta_E$ is a free algebra with the free-generating set $\{[a_1]\theta_E, \ldots, [a_n]\theta_E\}$, and $\mathcal{F}^n_{EI}/\theta_I$ is a free algebra with the free-generating set $\{[a_1]\theta_I, \ldots, [a_n]\theta_I\}$. From Theorem 1 [3]: $\mathcal{F}^n_{EI} \cong \mathcal{F}^n_{EI}/\theta_E \times \mathcal{F}^n_{EI}/\theta_I$, where $\mathcal{F}^n_{EI}/\theta_E \in E$, $\mathcal{F}^n_{EI}/\theta_I \in I$. Let be the 1-1 mapping of the generating set of $\mathcal{F}^n_{EI}$ onto a set of free generators of $\mathcal{F}^n_E$ and let f_h extend this mapping to a homomorphism. Let the symbol $Ker(f_h)$ denote kernel of this homomorphism. Since $\mathcal{F}^n_{EI}/Ker(f_h) \cong \mathcal{F}^n_E$, $\mathcal{F}^n_{EI}/Ker(f_h)$ has an exactly n-elementary set of generators. It is obvious that $\mathcal{F}^n_{EI}/Ker(f_h) \in E$, so from the minimal property of $\theta_E : \theta_E \subseteq Ker(f_h)$ and $\mathcal{F}^n_{EI}/\theta_E$ has an exactly n-elementary generating family. This means that $\mathcal{F}^n_{EI}/\theta_E/\theta_E \cong \mathcal{F}^n_E$. By similar arguments we prove that $\mathcal{F}^n_{EI}/\theta_I \cong \mathcal{F}^n_I$ and this complete the proof of the theorem.

Since every E- and every I-algebra with finite set of generators are finite, from Theorem 4 we have:

Corollary 4. *Every finitely generated EI-algebra is finite.*

References

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Department of Logic
Jagiellonian University