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Wiesław Dziobiak

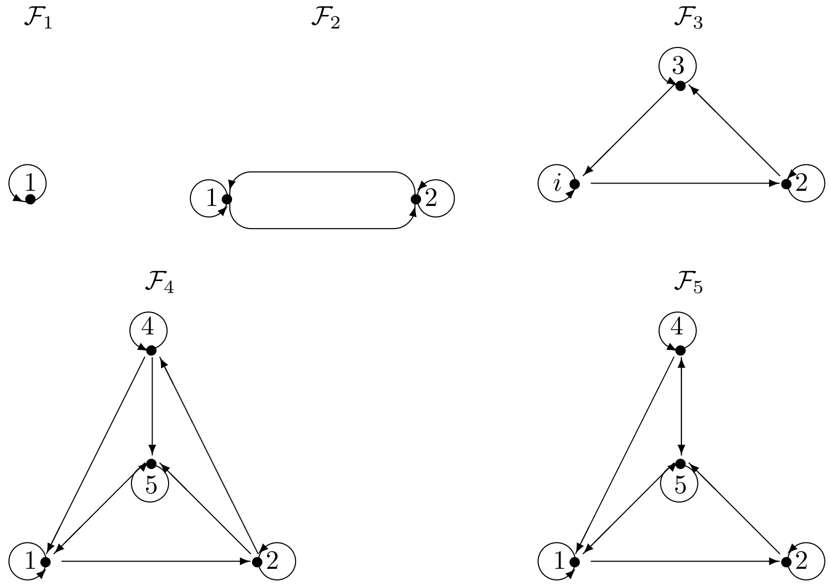
AN EXAMPLE CONCERNING THE LATTICE OF THE STRUCTURAL CONSEQUENCE OPERATIONS

This is an abstract of a lecture read at the Autumn School on Strongly Finite Sentential Calculi organized by Section of Logic, Polish Academy of Sciences, Institute of Philosophy and Sociology, Ustronie (Poland), November 1978.

Let $\mathcal{C}$ be the set of all structural consequence operations defined on some fixed sentential language $\underline{L}$. We know (see [5]) that $(\mathcal{C}, \leqslant)$ is a complete lattice, where the ordering $\leqslant$ is defined as follows: for each C_1, C_2 from $\mathcal{C}$, $C_1 \leqslant C_2$ iff $C_1(X) \subseteq C_2(X)$ for all $X \subseteq L$. Let SF^* denote the subset of $\mathcal{C}$ containing all strongly finite consequence operations (see [4]) for which there are strongly adequate matrices of degree 1 (see [5]). Our aim is to show that SF need not be closed under sup and inf in $(\mathcal{C}, \leqslant)$.

From now on, let $\underline{L} = (L, \vee, \wedge, \neg, \Box)$ be the sentential language of the type $\langle 2, 2, 1, 1\rangle$. A modal algebra is an algebra $\mathcal{A} = (A, +, \cdot, -, \ ^o)$ similar to $\underline{L}$ and satisfying: (i) $(A, +, \cdot, -)$ is Boolean algebra, (ii) $1^o = 1$ (1 is Boolean unit) and (ii) $(x \cdot y)^o = x^o \cdot y^o$ for all $x, y \in A$. It is known that from every frame $\mathcal{F} = (W, R)$, i.e. W is a non-empty set and $R \subseteq W \times W$, one can build the modal algebra $\mathcal{A} = (P(W), \cup, \cap, -, \ ^o)$ where $A^o = \{x \in W; \forall y(xRy \to y \in A)\}$ for all $A \in P(W)$.

Let us consider the modal algebras $\mathcal{A}_i$ $(i = 1, 2, 3, 4, 5)$ which are build from the frames $\mathcal{F}_i$ $(i = 1, 2, 3, 4, 5)$, given by the following diagrams, respectively.



("$x\cdot \rightarrow \cdot y$" means "$xRy$")

For these algebras one can prove the following lemma

LEMMA 1. *i)* $IS(\{\mathcal{A}_i\}) = I(\{\mathcal{A}_i, \mathcal{A}_3, \mathcal{A}_2, \mathcal{A}_1)$ *for* $i = 4, 5$
ii) $\mathcal{A}_4$ *and* $\mathcal{A}_5$ *are not isomorphic.*

Each variety of modal algebras is congruence distributive and all algebras $\mathcal{A}_i$ are simple. Then, by the above Lemma 1 and Corollary 3.4 from [1] we obtain the following

COROLLARY. $HSP(\{\mathcal{A}_4\}) \cap HSP(\{\mathcal{A}_5\}) = HSP(\{\mathcal{A}_3, \mathcal{A}_2, \mathcal{A}_1\})$

For each $\mathcal{A}_i$ let Cn_i $(i = 1, 2, 3, 4, 5)$ denote the consequence operation on $\underline{L}$ determined by the matrix $(\mathcal{A}_i, \{1\})$.

PROPOSITION 1. *i)* $sup\{Cn_4, Cn_5\} = inf\{Cn_2, Cn_3\}$
ii) $sup\{Cn_4, Cn_5\} \nleq Cn_2, Cn_3$

PROOF. (i): First we prove $(*) sup\{Cn_4, Cn_5\}(\emptyset) = inf\{Cn_2, Cn_3\}(\emptyset)$. Lemma 1(i) implies $\subseteq$. Since $sup\{Cn_4, Cn_5\}(\emptyset)$ is a normal modal logic which contains logics defined by the matrices $(\mathcal{A}_4, \{1\})$ and $(\mathcal{A}_5, \{1\})$ then the variety of modal algebras determined by it is contained in $HSP(\{\mathcal{A}_4\}) \cap HSP(\{\mathcal{A}_5\})$. Hence, by Corollary, we have $inf\{Cn_1, Cn_2, Cn_3\}(\emptyset) \subseteq sup\{Cn_4, Cn_5\}(\emptyset)$, but $\mathcal{A}_1 \in IS(\{\mathcal{A}_i\})$ $(i = 2, 3)$ and so the inclusion $\supseteq$ holds.

By virtue of Lemma 1(i), we prove only that $inf\{Cn_2, Cn_3\} \leqslant sup\{Cn_4, Cn_5\}$. Take any $\alpha \in Cn_i(X)$ $(i = 2, 3)$. Since $\mathcal{A}_2$ and $\mathcal{A}_3$ are finite then by Theorem 3 of [2] we have $\alpha \in Cn_i(X_f)$ $(i = 2, 3)$ for some finite $X_f \subseteq X$. Hence, $\Box^2 \bigwedge X_f \to \alpha \in Cn_i(\emptyset)$ for $i = 1, 2$ ($\bigwedge X_f$ denotes conjunction of all formulas from X_f). This and $(*)$ give us that $\alpha \in sup\{Cn_4, Cn_5\}(X)$, which completes the proof of (i).

(ii): Follows from (i) and from $Cn(\emptyset) \subsetneq Cn_3(\emptyset)$ and $Cn_3(\emptyset) \subsetneq Cn_2(\emptyset)$ (see [3]). QED

Notice that Proposition 1(i) implies that $sup\{Cn_4, Cn_5\}$ is a strongly finite consequence operation on $\underline{L}$.

It is easy to prove

LEMMA 3. *Let $\mathcal{A} = (A, +, \cdot, -, {}^o)$ be an algebra similar to $\underline{L}$ and $(\mathcal{A}, D)$ be a matrix of degree 1 satysfying: (i) all classical tautologies and the formula $\Box(p \to q) \to (\Box p \to \Box q)$ are valid in $(\mathcal{A}, D)$, (ii) the modus ponens for $\to$ and the necessitation rule are rules of the consequence operation $Cn_{(\mathcal{A},D)}$ determined by $(\mathcal{A}, D)$. Then,*

(1) the relation $\equiv_D$ defined $x \equiv_D y$ iff $(-x + y) \cdot (-y + x) \in D$ is a congruence relation on $\mathcal{A}$.

(2) $\mathcal{A}/\equiv_D$ is a modal algebra in which D is Boolean unit

(3) $Cn_{(\mathcal{A},D)} = Cn_{(\mathcal{A}/\equiv_D, \{D\})}$.

PROPOSITION 2. *There is no matrix of degree 1 strongly adequate for $sup\{Cn_4, Cn_5\}$, as well as for $inf\{Cn_2, Cn_3\}$.*

PROOF. By virtue of Proposition 1 it is enough to settle this for *sup*. Suppose that $sup\{Cn_4, Cn_5\}$ has a strongly adequate matrix of degree 1. Then, by Lemma 3 we can assume that $sup\{Cn_4, Cn_5\} = Cn_{(\mathcal{A},\{1\})}$ for modal algebra $\mathcal{A}$. Then, by Proposition 1, we obtain $\mathcal{A} \in HSP(\{\mathcal{A}_2, \mathcal{A}_3\})$. Hence and from Birkhoff's Theorem there are subdirectly irreducible algebras $\mathcal{A}_t$, $t \in T$, belonging to $HSP(\{\mathcal{A}_2, \mathcal{A}_3\})$ such that $\mathcal{A}$ is isomorphic to some subdirect product of them. Therefore, $Cn_{(\mathcal{A},\{1\})} \geqslant Cn_{(\prod\{\mathcal{A}_t; t\in T\},\{1\})}$. Notice that the only non-trivial and non-isomorphic subdirectly irreducible modal algebras from $HSP(\{\mathcal{A}_2, \mathcal{A}_3\})$ are $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$. Then, by Theorem 1 of [6] and Proposition 1(ii), we have that the consequence operation determined on $\underline{L}$ by $(\prod\{\mathcal{A}_t; t \in T\}, \{1\})$ must be equal to one of the following: $Cn_{(\mathcal{A}_1\times\mathcal{A}_2,\{1\})}$, $Cn_{(\mathcal{A}_1\times\mathcal{A}_3,\{1\})}$, $Cn_{(\mathcal{A}_2\times\mathcal{A}_3,\{1\})}$ and $Cn_{(\mathcal{A}_1\times\mathcal{A}_2\times\mathcal{A}_3,\{1\})}$. Let α denote the formula $(p \wedge \neg\Box p) \vee (q \wedge \neg\Box q)$. It is easy to check that $Cn_1(\alpha) = L$, $Cn_2(\alpha) \neq L$ and $Cn_3(\alpha) = L$. All matrices $(\mathcal{A}_i, \{1\})$ $(i = 1, 2, 3)$ are proper in the sense of (6). Hence $Cn_{(\mathcal{A}_1\times\mathcal{A}_2,\{1\})}(\alpha) = Cn_{(\mathcal{A}_1,\mathcal{A}_3,\{1\})}(\alpha) = Cn_{(\mathcal{A}_2\times\mathcal{A}_3,\{1\})}(\alpha) = Cn_{(\mathcal{A}_1\times\mathcal{A}_2\times\mathcal{A}_2,\{1\})}(\alpha) = L$, but, by Lemma 1(i) and from our assumption, we have $Cn_{(\mathcal{A},\{1\})}(\alpha) \neq L$, which is impossible. QED

From the above considerations we obtain

(1) SF^* need not be closed under *sup* and inf in $(\mathcal{C}, \leqslant)$ also, we get

(2) The set of all consequence operations belonging to $\mathcal{C}$ and being uniform (see [2]) need not be closed under *sup* and inf in $(\mathcal{C}, \leqslant)$

(3) The problem of supremum of finite set of strongly finite consequence operations on $\underline{L}$ which is stated in [4] has negative answer in the subset of $\mathcal{C}$ consisting of all consequence operations having finite strongly adequate matrices of degree 1.

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Institute of Mathematics
N. Copernicus University
Toruń