

Jan Zygmunt

## ENTAILMENT RELATIONS AND MATRICES I

This is an abstract of a paper presented to the Logic Seminar of the Section of Logic in Wrocław, December 1978, February 1979.

In this paper we extend some techniques from the theory of consequences operations and logical matrices (elaborated by J. Łoś, R. Suszko and R. Wójcicki) into the wider area of entailment relations.

Let  $\underline{L}$  be a fixed sentential language with finitary connectives and infinite number of sentential variables  $Var(\underline{L}) = \{p_1, p_2, \dots\}$ . By  $\underline{L}^{(n)}$  we shall denote the subalgebra of  $\underline{L}$  generated by  $\{p_1, \dots, p_n\}$ . The set of formulas of  $\underline{L}$  will be denoted by  $L$ .

Any relation  $\vdash \subseteq P(L) \times P(L)$  is said to be an *entailment relation* (cf. [3], p. 29 and [2]) iff it satisfies the following conditions ( $X, Y$  are arbitrary subsets of  $L$ ):

- (R) If  $X \cap Y \neq Y$  then  $X \vdash Y$ .
- (M) If  $X \vdash Y$ , where  $X' \supseteq X$ ,  $Y' \supseteq Y$ , then  $X' \vdash Y'$ .
- (C) If for all  $Z \subseteq L$   $X \cup Z \vdash Y \cup L \setminus Z$  then  $X \vdash Y$ .

If moreover  $\vdash$  satisfies the following condition

- (S) If  $X \vdash Y$  then  $eX \vdash eY$ , all  $e \in Hom(\underline{L}, \underline{L})$

it will be called a *structural entailment relation* (cf. [1] for the concept of structural consequence operation).

Convention:  $X, Y \vdash Z, W$ , and the like, will be a shorthand version of  $X \cup Y \vdash Z \cup W$ .

We say that  $\vdash$  is *finitistic* iff whenever  $X \vdash Y$  there exists finite subsets  $X'$  and  $Y'$  of  $X$  and  $Y$  such that  $X' \vdash Y'$ . For finitistic entailment relations (C) is equivalent to  $(C_f)$ , where

$(C_f)$   $X \vdash \alpha, Y$  and  $X, \alpha \vdash Y$  implies  $X \vdash Y$ .

Let  $\xi$  (resp  $\xi^0$ ) denote the set of all entailment relations (structural entailment relations) on  $\underline{L}$ . Consider the following partially ordered sets:  $(\xi, \subseteq)$  and  $(\xi^0, \subseteq)$ . One can prove that the first one is a complete atomic Boolean algebra and the latter is a complete distributive sublattice of the former. Moreover, there exist natural embeddings of the lattice of consequence operations resp. the lattice of structural consequence operations on  $\underline{L}$  into  $(\xi, \subseteq)$  resp.  $(\xi^0, \subseteq)$ .

If  $\underline{A}$  is an algebra similar to  $\underline{L}$  and  $\underline{D}$  is a family of subsets of  $A$  then the pair  $M = (\underline{A}, \underline{D})$  will be said to be a (*logical*) *matrix* for  $\underline{L}$ . For any such  $M$  we define a relation on  $P(L)$  as follows:

$$X \Vdash_M \text{ iff } \forall_{h \in \text{Hom}(\underline{L}, \underline{A})} \forall_{D \in \underline{D}} [hX \subseteq D \Rightarrow hY \cap D \neq \emptyset].$$

PROPOSITION 1.  $\Vdash_M \in \xi^0$ .

PROPOSITION 2. For each  $\vdash \in \xi^0$  there exists a matrix  $M$  such that  $\vdash = \Vdash_M$ .

We say that  $\vdash$  satisfies  $(X^n)$  iff

$(*)^n$   $X \vdash Y$  iff  $eX \vdash eY$ , all  $e \in \text{Hom}(\underline{L}, \underline{L}^{(n)})$ .

For any natural  $n \geq 1$  and any  $\vdash$  we define the following Lindenbaum-type matrix:

$$M_{\vdash}^{(n)} = (\underline{L}^{(n)}, \{Z \cap L^{(n)} : Z \not\vdash L - Z, Z \subseteq L\})$$

LEMMA 3. If  $\vdash \in \xi$  satisfies  $(*)^n$  then  $\vdash = \Vdash_M$ , where  $M = M_{\vdash}^{(n)}$ .

We say that an  $\vdash \in \xi$  is *strongly finite* (cf. [3]) iff there exist a finite matrix  $M$  such that  $\vdash = \Vdash_M$ .

Choosing one of the standard methods, one can prove that  $\Vdash_M$  is finitistic whenever  $M$  is a finite matrix.

PROPOSITION 4 (Criterion of strong finiteness, cf. [3], p. 58). Any  $\vdash \in \xi^0$  is strongly finite iff there exists a natural number  $n \geq 1$  such that

- (i)  $\vdash$  satisfies  $(*)^n$ ,
- (ii) there exists a congruence  $\theta$  of  $\underline{L}$  such that for all  $\alpha, \beta \in L$  if  $\alpha\theta\beta$  then  $\alpha \vdash \beta$  and  $\underline{L}^{(n)}/\theta$  is finite.

Let  $A \subseteq L$  and  $B \subseteq L$ . By the  $(A, B)$ -strengthening of  $\vdash$  we mean the least entailment relation on  $\underline{L}$  containing  $\vdash$  and the following two sets:

$$\begin{aligned} E(A) &= \{(\emptyset, \{e\alpha\}) : \alpha \in A, e \in \text{End}(\underline{L})\} \\ E(B) &= \{(\{e\beta\}, \emptyset) : \beta \in B, e \in \text{End}(\underline{L})\}. \end{aligned}$$

Let us denote by  $\vdash_B^A$  the  $(A, B)$ -strengthening of  $\vdash$ . We have:

LEMMA 5.  $X \vdash_B^A Y$  iff  $X, Sb(A) \vdash X, Sb(B)$ .

THEOREM 6. Any  $(A, B)$ -strengthening of a strongly finite entailment relation is strongly finite.

THEOREM 7. The number of  $(A, B)$ -strengthenings of a strongly finite entailment relation is finite.

THEOREM 8. For any finite matrices  $M$  and  $N$  there is an effective procedure to determine whether a)  $\Vdash_M = \Vdash_N$ , b)  $\Vdash_M \subseteq \Vdash_N$ , c)  $\Vdash_N \subseteq \Vdash_M$ , or d)  $\Vdash_M$  and  $\Vdash_N$  are not comparable.

COROLLARY 9. For any finite matrix  $M$  there is an effective procedure to describe the diagram of the following partially ordered set:  $\{\vdash : \vdash_M \subseteq \vdash\}$ , where  $\vdash$  is an  $(A, B)$ -strengthening of  $\vdash$ .

AUTHOR'S ACKNOWLEDGMENT. The author is very indebted to his colleague Jacek Hawranek for many conversations on the subject of this paper.

## References

- [1] J. Łoś and R. Suszko, *Remarks on sentential logics*, **Indagationes Mathematicae** 20 (1958), pp. 177–183.
- [2] D. Scott, *Completeness and axiomatizability in many valued logic*. *Proceedings of the Tarski Symposium*, ed. by L. Henkin et al., **Proc. of Symposia in Pure Math.**, vol. 25 (1974), pp. 441–435.

[3] D. J. Shoesmith and T. J. Smiley, **Multiple-conclusion logic**, Cambridge University Press, 1978.

[4] R. Wójcicki, *Strongly finite sentential calculi*, **Selected Papers on Łukasiewicz Sentential Calculi**, ed. by R. Wójcicki and G. Malinowski, Wrocław, 1977, pp. 53–77.

*The Section of Logic  
Institute of Philosophy and Sociology  
Polish Academy of Sciences*