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ON TWO NOTIONS CONCERNING THE STRUCTURAL SENTENTIAL CALCULI

In this note we investigate the notions of reflection and projection of structural sentential calculi as introduced by R. Wójcicki (see II §2 in [4]). This note was read at the Autumn School on Strongly Finite Sentential Calculi organized by Section of Logic, Polish Academy of Sciences, Institute of Philosophy and Sociology, Ustronie (Poland), November 1978.

For any structural sentential calculus $(\underline{L}, C)$ and sentential language $\underline{L}$, similar to $\underline{L}$, we put $\Theta_C^{L_o} = \{C' | C'$ is a structural consequence operation on $\underline{L}_o$ satisfying: $A \in C(X)$ only if $eA \in C'(eX)$, all $X \cup \{A\} \subseteq L$ and $e \in Hom(\underline{L}, \underline{L}_o)\}$. Let us recall the following

DEFINITION (R. Wójciciki, see III §in [4]). Let $(\underline{L}, C)$ be structural and $\underline{L}_o$ be a sentential language similar to $\underline{L}$. The *projection of* $(\underline{L}, C)$ *onto* $\underline{L}_o$ is the sentential calculus $(\underline{L}_o, C|L_o)$, where $C|L_o = inf\,\Theta_C^{L_o}$. The *reflection of* $(\underline{L}, C)$ *form* $\underline{L}_o$ *is* $(\underline{L}, L_o|C)$ where $L_o|C = inf\,\Theta_{C|L_o}^L$.

For a sentential language L, by $V(\underline{L})$ we denote the set of all sentential variables of $\underline{L}$.

LEMMA 1. *Let* $(\underline{L}, C)$ *be structural such that* $card V(\underline{L}) \geqslant \aleph_0$ *and let* $\underline{L}_o$ *be a sentential language similar to* $\underline{L}$. *Define the mapping* $C_o : P(L_o) \to P(L_o)$ *by:* $A \in C_o(X)$ *iff there are* $Y \cup \{B\} \subseteq L$, $e \in Hom(\underline{L}, \underline{L}_o)$ *such that* $eB = A$, $eY \subseteq X$ *and* $B \in C(Y)$, *for all* $X \subseteq L_o$ *and* $A \in L_o$. *Then* C_o *is a structural consequence operation on* $\underline{L}_o$.

For $\underline{L}_o$ being normal extension of $\underline{L}$, i.e. $V(\underline{L}) \subseteq V(\underline{L}_o)$, this lemma was proved in [5] (see the proof of th. 1, §1 in [5]).

From the Lemma 1 it follows

COROLLARY 2. *The consequence operation defined in Lemma 1 is equal to the projection of $(\underline{L}, C)$ onto $\underline{L}_o$. Moreover, the cardinality of $C|L_o$ is smaller than the cardinality of C.*

Let the assumptions of Lemma 1 hold. Then, one can show by aid of Lemma 1 that the sentential calculus $(\underline{L}_o, C_o)$ (C_o is defined in Lemma 1) is inductively generated from $(\underline{L}, C)$ in the sense of [2].

The following theorem was stated by R. Wójcicki (see III §2 in [4]).

THEOREM 3. *(i) Let $(\underline{L}, C)$ be structural such that card $\leqslant cardV(\underline{L})$ and $\aleph_0 \leqslant cardV(\underline{L})$, while $\underline{L}_o$ is a normal extension of $\underline{L}$. Then $C|L_o$ is a natural extension of C in the sense of [3], i.e. the following condition holds: $A \in (C|L_o)(X)$ iff there are $Y \subseteq X$ and $e \in Aut(\underline{L}_o)$ such that $cardY < cardC$, $eA \in L$, $eY \subseteq L$ and $eA \in C(eY)$, for all $X \subseteq L$ and $A \in L$.*

(ii) Let $(\underline{L}, C)$ be structural and $\underline{L}_o$ be a sentential language similar to $\underline{L}$ and such that $\underline{L}$ is a normal extension of $\underline{L}_o$. Then $(C|L_o)(X) = C(X) \cap L_o$, for all $X \subseteq L_o$.

PROOF. (i): Use Lemma 1. (ii): if card $V(\underline{L}_o) < \aleph_0$ then (ii) is trivial, otherwise Lemma 1 suffices, too.

Let $\underline{L}$ be a sentential language such that $V(\underline{L}) = \{x_\gamma | \gamma < \alpha\}$ for some ordinal number α. The sublanguage of $\underline{L}$ generated by $\{x_\gamma | \gamma < \beta\}(\beta \leqslant \alpha)$ will be denoted by $\underline{L}^{(\beta)}$. Let $(\underline{L}, C)$ be structural and $\underline{L}^{(\beta)}$ be a sublanguage of $\underline{L}$. To simplify, denote the projection of $(\underline{L}, C)$ onto $\underline{L}^{(\beta)}$ by $(\underline{L}^{(\beta)}, C|\beta)$, and the reflection of $(\underline{L}, C)$ from $\underline{L}^{(\beta)}$ by $(\underline{L}, \beta|C)$.

Let r be a rule of inference on the sentential language $\underline{L}$. We say that r is a *sequential rule of order β* iff there are $\alpha \in L^{(\beta)}$ and $X \subseteq L^{(\beta)}$, such that r is generated by the sequent of the form $\langle X, \alpha \rangle$. We say that a structural consequence operation C on $\underline{L}$ has a *sequential basis of order β* iff there exists a set Q satisfying $C = Cn_Q$, consisting of sequential inference rules of order β.

THEOREM 4. *Let $(\underline{L}, C)$ be structural such that $V(\underline{L}) = \{x_\gamma | \gamma < \alpha\}$ and let β be a limit ordinal number smaller than α. Then the following conditions are equivalent*

(I) $sup\{\gamma|C; 1 \leqslant \gamma < \beta\} = C$

(II) C has a sequential basis of order β.

This theorem was suggested by R. Wójcicki (see II §2 in [4]).

COROLLARY 5. *Let $(\underline{L}, C)$ be a standard sentential calculus and $V(\underline{L}) = \{x_\gamma; \gamma < \omega\}$. Then $sup\{k|C; 1 \leqslant k < \omega\} = C$.*

PROOF. Use th. 3 of [3], the above Theorem 4 and notice that $1|C \leqslant 2|C \leqslant \ldots \leqslant k|C \leqslant \ldots \leqslant C$.

EXAMPLES. All sentential language from the examples listed below are standard, i.e. their sets of sentential variables are denumerable but infinite.

1. $(\underline{L}, C_{INT})$, C_{INT} is the consequence operation defined on the full language $\underline{L}$ by the set of all theorem of the Heyting's intuitionistic logic and the modus ponens rule. For it we have that $3|C_{INT} = C_{INT}$.

2. $(\underline{L}, C_{Ł_n h})$ $(n \geqslant 2)$, $C_{Ł_n}$ is the consequence operation defined on the full language $\underline{L}$ by the set of all theorem of Łukasiewicz's n-valued logic and the rule of modus ponens. The method of axiomatization of Łukasiewicz's logics due to Tokarz [6] gives that $3|C_{Ł_n} = C_{Ł_n}$.

3. $(\underline{L}, Cn_{\mathcal{A} \times \mathcal{B}})$ (see Wroński [10]), $\underline{L}$ is a sentential language of type $\langle 2 \rangle$ and $Cn_{\mathcal{A} \times \mathcal{B}}$ is the matrix consequence operation on $\underline{L}$ determined by the direct product of matrices $\mathcal{A} = ((\{0, 1, 2\}, \cdot), \{0\})$ and $\mathcal{B} = ((\{0, 2\}, \cdot), \{0\})$ where the binary operation $\cdot$ is defined as follows: $0 \cdot 0 = 2 \cdot 2 = 2, 1 \cdot 1 = 1$ and $a \cdot b = 0$ otherwise. By Lemma 1.1 of [10] one can show $k|Cn_{\mathcal{A} \times \mathcal{B}} \nleqslant Cn_{\mathcal{A} \times \mathcal{B}}$ is not finitely based which was proved in Wroński [10].

4. $(\underline{L}, Cn_{\mathcal{A}})$ (see Urquhart [8]), $\underline{L}$ is as in example 3 while $Cn_{\mathcal{A}}$ is gives on $\underline{L}$ by the matrix $\mathcal{A} = ((\{0, 1, 2, 3, 4\}, \cdot), 0)$, where the binary operation $\cdot$ is defined by setting $0 \cdot 2 = 0 \cdot 3 = 1 \cdot 3 = 0, 1 \cdot 2 = 1$ and $a \cdot b = 4$ otherwise. By Lemma 3 of [8] we see that $k|Cn_{\mathcal{A}} \nleqslant Cn_{\mathcal{A}}$ for all $k < \omega$, and so $Cn_{\mathcal{A}}$ is not finitely based which was proved in Urquhart [8].

5. $(\underline{L}, Cn_{\mathcal{A}_2 \times \mathcal{A}_1})$ (see Tokarz [7]), $\underline{L}$ is sentential language of type $\langle 1, 2 \rangle$ while $Cn_{\mathcal{A}_2 \times \mathcal{A}_1}$ is the consequence operation defined on $\underline{L}$ by the direct product of 4 and 2-elements Sugihara matrices, respectively. Applying lemmas A and C from [7] one can prove that $k|Cn_{\mathcal{A}_2 \times \mathcal{A}_1} \nleqslant Cn_{\mathcal{A}_2 \times \mathcal{A}_1}$. Furthermore, similarly to example 3 one may show that $Cn_{\mathcal{A}_2 \times \mathcal{A}_1}$ is not finitely based.

By virtue of the examples (3) – (5) one may pose the following ques-

tion: is there a strongly finite consequence operation C on some standard sentential language having the properties (i) $k|C = C$ for some $k < \omega$, (ii) C is not finitely based?

The next theorem pours some light upon this question.

THEOREM 6. *Let $(\underline{L}, C)$ be a strongly finite sentential calculus ($\underline{L}$-standard) with implication, in the sense of Rasiowa (see [9]). Moreover, let $C(\emptyset) = Cn_Q(\emptyset)$ for some finite set Q of standard rules of inference such that $Cn_Q \leqslant C$. Then the following conditions are equivalent*

(I) $k|C = C$ *for some* $k < \omega$

(II) C *is finitely based.*

This theorem permits to settle the following

COROLLARY 7. *Every strengthening of $(\underline{L}, C_{\underline{L}_n})$ ($n \geqslant 2$) is finitely based.*

We end our disscusion with the following theorem, to be proved by applying Lemma 1.

THEOREM 8. *Let $(\underline{L}, C)$ be a structural sentential calculus such that $card C = 2^{\aleph_0}$. Then on the base $ZF + AC$ the following conditions are equivalent*

(I) $\omega|C = C$

(II) *continuum hypothesis.*

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