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FINITE STRUCTURAL AXIOMATIZATION OF EVERY FINITE-VALUED PROPOSITIONAL CALCULUS

In [2] A. Wroński proved that there is a strongly finite consequence C which is not finitely based, i.e., for every consequence C^+ determined by a finite set of standard rules $C \neq C^+$. In this paper it will be proved that for every strongly finite consequence C there is a consequence C^+ determined by a finite set of structural rules such that $C(\emptyset) = C^+(\emptyset)$ and $\overline{C} = \overline{C}^+$ (where $\overline{C}$, $\overline{C}^+$ are consequences obtained by adding to the rules of C, C^+ respectively the rule of substitution). Moreover it will be shown that by certain assumptions $C = C^+$.

Let $\underline{L} = (L, CON)$ be any absolutely free algebra with $VAR = \{p, p_1, \ldots\}$ as a set of generators, where CON is a finite sequence of operations denoted by sentential connectives. $\underline{L}^k = (L^k, CON)$ is a subalgebra of $\underline{L}$ generated by $p_1, \ldots, p_k$. The letter r denotes any rule and R denotes any set of rules. The rule of substitution is denoted by SUB. A rule r is structural iff for all $a \in L$, $X \subseteq L$ if $\langle X, a\rangle \in r$ then for every substitution $e : \underline{L} \to \underline{L}$ $\langle eX, ea\rangle \in r$. A mapping $C : 2^L \to 2^L$ is a consequence operation in $\underline{L}$ iff for every $X, Y \subseteq L$ $X \subseteq CC(X) \subseteq C(X \cup Y)$. R determined a consequence operation Cn_R, i.e., for every $X \subseteq L$ $Cn_R(X)$ is the smallest subset of L containing X and closed under the rules of R. For every $X \subseteq L$ $SUB(X) = Cn_{\{SUB\}}(X)$. A consequence C is structural iff for every substitution $e : \underline{L} \to \underline{L}$ and every $X \subseteq L$ $eC(X) \subseteq C(eX)$. The symbol $\overline{C}$ denotes a consequence obtained by adding to the rules of C the rule of substitution (i.e. if $C = Cn_R$ then $\overline{C} = Cn_{R\cup\{SUB\}}$. The symbol M denotes a generalized matrix (cf. [1]) associated with $\underline{L}$, and Cn_M denotes the matrix consequence determined by M. A consequence C is strongly finite iff there is a finite matrix M such that $C = Cn_M$. Instead of $C(\{a_1,\ldots,a_i\})$ we write $C(a_1,\ldots,a_i)$ and we write $C_1 \leqslant C_2$ instead of: for every $X \subseteq L$ $C_1(X) \subseteq C_2(X)$.

In this paper we assume that M is any fixed k-valued generalized matrix. h is any valuation in this matrix. $a \sim_M b$ iff for every valuation h $ha = hb$. Instead of $|a| \sim_M$, $L^k/\sim_M$ we write a_M, L^k/M respectively.

Let C be a structural consequence, R a set of structural rules, $e : \underline{L} \to \underline{L}$ a substitution and $a \in L$, $X \subseteq L$. Then the following theorems (see for example [1]) hold:

I. the set L^k/M is finite,
II. Cn_R, Cn_M are structural consequences,
III. each strongly finite consequence is finite,
IV. if $a \in C(SUB(X))$ then $ea \in C(SUB(X))$,
V. $\overline{C}(X) = C(SUB(X))$.

G is a rule of the form $\{\langle\{e(a(p_i/p_j)) : 1 \leqslant i \neq j \leqslant k+1\}, ea\rangle : p_1,\ldots,p_{k+1} \in VAR,\ a \in L,\ e : \underline{L} \to \underline{L}$ is any substitution$\}$. This rule is similar to Wroński's rule r_n in [3]. $MIN(X)$ is the set of all formulas of X having minimal length (a variable $p \in VAR$ has length equal to 1, and if f is any i-argument connective of CON, $a_1,\ldots,a_i \in L$ have lengths equal to $m_1,\ldots,m_i$ respectively, then $f(a_1,\ldots,a_i)$ have length equal to $m_1 + \ldots + m_i + 1$).

$S(a) = MIN(a_M \cap L^k)$

$U = \bigcup_{a \in L^k} S(a)$

$T(a) = \{f(a_1,\ldots,a_i) : a_1,\ldots,a_i \in U,\ f$ is any i-argument connective of CON, $a_M = (f(a_1,\ldots,a_i))_M\}$.

$W(a) = S(a) \times S(a) \cup S(a) \times T(a)$

$REX(a)$ is the set of all rules of the form $\{\langle e(b(p/c)), e(b(p/d))\rangle : p \in VAR,\ b \in L,\ e : \underline{L} \to \underline{L}$ is any substitution where $\langle c,d\rangle \in W(a)$ or $\langle d,c\rangle \in W(a)$.

$REX = \bigcup_{a \in L^k} REX(a)$

$RVAL$ is the set of all rules of the form $\{\langle eX, ea\rangle : e : \underline{L} \to \underline{L}$ is any substitution$\}$ where $a \in U \cap Cn_M(X)$ and $X \subseteq U$.

RAX is the set of all rules of the form $\{\langle\emptyset, ea\rangle : e : \underline{L} \to \underline{L}$ is any substitution where $a \in U \cap Cn_M(\emptyset)$.

$RLS = RAX \cup REX \cup RVAL \cup \{G\}$
$C_M = Cn_{RLS}$

LEMMA 1.

(i) RLS *is finite,*
(ii) C_M *is structural,*
(iii) $C_M \leqslant Cn_M$.

PROOF. It is obvious that $S(a)$ is finite. Notice that if $a_M = b_M$ then $S(a) = S(b)$. By I L^k/M is finite. Then U is finite and consequently $T(a)$ is finite. Then $W(a)$ is finite and consequently $REX(a)$, REX are finite. Then (i) holds. It is easy to see that every rule of RLS is structural. Hence by II (ii) holds. Let h be any valuation and $e : \underline{L} \to \underline{L}$ any substitution. There are i, j such that $1 \leqslant i \neq j \leqslant k+1$ and $hep_i = hep_j$. Hence $he(a(p_i/p_j)) = hea$. So G is valid in M. It is easy to see that the other rules of RLS are valid in M. Q.E.D.

We write $a \vdash b$ instead of: b is derivable from a by using the rules of REX. Instead of $a \vdash b$ and $b \vdash a$ we write $a \dashv\vdash b$. a is a subformula of b iff $a = b$ or if $b = f(b_1, \ldots, , b_i)$ (f is any i-argument connective of CON), then there is $j \leqslant i$ such that a is a subformula of b_j.

LEMMA 2. *For all* $a, b \in L^k$ $a_M = b_M$ *iff* $a \dashv\vdash b$.

PROOF. First we prove that

1. if a $L^k - U$ then there is $b \in L^k$ such that $b \vdash a$ and b is shorter than a.

Suppose that $a \in L^k - U$. Hence there is a subformula b of a such that $b \in T(b) - S(b)$. Let c be a formula such that $c \in S(b)$. Then $\langle c, b\rangle \in S(b) \times T(b) \subseteq W(b)$. Let $d \in L$ be a formula such that $d(p/b) = a$ and $a \neq d$. Then $\langle d(p/c), a\rangle \in r \in REX(b) \subseteq REX$ and $d(p/c)$ is shorter than a. By 1 we obtain

2. if $a \in L^k$ then there is $b \in U$ such that $b \vdash a$.

One can prove that

3. for all $a, b \in L^k$ if $a \vdash b$ then $a_M = b_M$,
4. for all $a, b \in U$ if $a_M = b_M$ then $a \dashv\vdash b$,

5. if $a \vdash b$ then $b \vdash a$.

Suppose that $a, b \in L^k$ and $a_M = b_M$. By 2 there are $a_1, b_1 \in U$ such that $a_1 \vdash a$, $b_1 \vdash b$. By 3 $a_{1_M} = a_M$, $b_{1_M} = b_M$. Then $a_{1_M} = b_{1_M}$. By 4 $a_1 \dashv\vdash b_1$. By 5 $a \dashv\vdash b$. Conversely, by 3, 5 we obtain the second part of the proof. Q.E.D.

LEMMA 3. *For all $a \in L$, $X \subseteq L$ if $a \in Cn_M(X)$ then for every substitution $e : \underline{L} \to \underline{L}^k$ $ea \in C_M(eX)$.*

PROOF. Suppose that $a \in Cn_M(X)$ and $e : \underline{L} \to \underline{L}^k$ is a substitution. By III there are $a_1, \ldots, a_i \in X$ such that $a \in Cn_M(a_1, \ldots, a_i)$. By II Cn_M is structural, then $ea \in Cn_M(ea_1, \ldots, ea_i)$. There are $b, b_1, \ldots, b_i \in U$ such that $b_M = (ea)_M$, $b_{1_M} = (ea_1)_M, \ldots, b_{i_M} = (ea_i)_M$. Then $b \in Cn_M(b_1, \ldots, b_i)$, and $\langle b_1, \ldots, b_i, b \rangle \in r \in RVAL$. Hence $b \in C_M(b_1, \ldots, b_i)$. By Lemma 2 $ea \in C_M(ea_1, \ldots, ea_i) \subseteq C_M(eX)$. Q.E.D.

The symbol $VAR(X)$ denotes the set of all variables of formulas belonging to $X \subseteq L$. The symbol $P(\{a\})$ denotes the set $\{a(p_i/p_j) : p_i, p_j \in VAR(\{a\}), i \neq j$. Let $P(X) = P^1(X) = \bigcup_{a \in X} P(\{a\})$ and for $i > 1$ $P^i(X) = P(P^{i-1}(X))$. It can be proved that

PROPOSITION. *If $\overline{\overline{VAR(a)}} = j > k$ and $j - k = i$, then $a \in C_M(P^i(\{a\}))$, every formula $b \in P^i(\{a\})$ contains exactly k variables and there is a substitution $e : \underline{L} \to \underline{L}$ such that $b = ea$.*

LEMMA 4. *For all $a \in L$, $X \subseteq L$, if for every substitution $e : \underline{L} \to \underline{L}^k$ $ea \in C_M(SUB(X))$ then $a \in C_M(SUB(X))$.*

PROOF. Let $p_{m_1}, \ldots, p_{m_j}$ be all variables of a. (If $j \leqslant k$ then by IV the proof is simple). Let $j > k$, $j - k = i$. These assumptions are all assumptions of Proposition. Then it is sufficient to prove that $b \in C_M(SUB(X))$. Let $p_{n_1}, \ldots, p_{n_k}$ be all variables of b. Let $e_1 : \underline{L} \to \underline{L}^k$ be a assumption such that $e_1 p_{n_1} = p_1, \ldots, e_1 p_{n_k} = p_k$. Then by the assumption of the lemma $e_1 b = e_1 ea \in C_M(SUB(X))$. Let $e_2 : \underline{L} \to \underline{L}$ be a substitution such that $e_2 p_1 = p_{n_1}, \ldots, e_2 p_k = e_2 p_{n_k}$. Then by IV $b = e_2 e_1 b \in C_M(SUB(X))$. Q.E.D.

By Lemmas 1, 4, 3 and II, V we obtain

THEOREM. *The consequence C_M is determined by the finite set of struc-*

tural rules (RLS) and the following equations hold:

(i) $C_M(\emptyset) = Cn_M(\emptyset)$
(ii) $\overline{C_M} = \overline{Cn_M}$

COROLLARY. *If a connective* $\to$*, such that for all* $a, b \in L$*,* $X \subseteq L$*,* $a \in Cn_M(X \cup \{b\})$ *iff* $b \to a \in Cn_M(X)$*, can be defined in* $\underline{L}$ *then* $C_M = Cn_M$*.*

PROOF. Suppose that $a \in Cn_M(X)$. By III there are $a_1, \ldots, a_i \in X$ such that $a \in Cn_M(a_1, \ldots, a_i)$. Then $a_1 \to (a_2 \to \ldots \to (a_i \to a) \ldots) \in Cn_M(\emptyset)$. By Theorem (i) $a_1 \to (a_2 \to \ldots \to (a_i \to a) \ldots) \in C_M(\emptyset)$. If $k \leqslant 2$ (for $k = 1$ the proof is trivial) then the rule of Modus Ponens for $\to$ by the assumption belongs to $RVAL$. Then $a \in C_M(a_1, \ldots, a_i) \subseteq C_M(X)$. Hence by Lemma 1 (iii) $C_M = Cn_M$. Q.E.D.

References

[1] R. Wójcicki, *Strongly finite sentential calculi*, [in:] **Selected papers on Łukasiewicz sentential calculi**, Wrocław 1977.

[2] A. Wroński, *On finitely based consequence operations*, **Studia Logica**, XXXV (1976), pp. 453–458.

[3] A. Wroński, *On the depth of a consequence operation*, **Bulletin of the Section of Logic**, vol. 6 (1977), No. 3, pp. 96–101.

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