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A THREE ELEMENT MATRIX WHOSE CONSEQUENCE OPERATION IS NOT FINITELY BASED

The question whether the consequence operation of a finite matrix is always finitely based was proposed by S. L. Bloom [1] and also by R. Wójcicki [3]. The negative answer – a six-element matrix – was given in [4] and next a five-element matrix was found by A. Urquhart [2]. In this paper we will show that no finite basis exists for the consequence operation $C_{\mathbb{M}}$ of the matrix $\mathbb{M} = \langle\langle\{0, 1, 2\}, \cdot\rangle, \{2\}\rangle$ where $\cdot$ is a binary operation such that $0 \cdot 1 = 0$ and $a \cdot b = 2$ otherwise. Our proof is a slight modification of that given by A. Urquhart in [2] and most of the notions and techniques used here are borrowed from [2]. Our result, however, is the best possible because every consequence operation of a two element matrix is finitely based (oral information of Prof. Wolfgang Rautenberg).

Let the symbol V denotes an infinite set of variables and the symbol F the set of formulas constructed as usual by means of variables from V and one binary connective $\cdot$. We adopt the convention of associating to the left and ignoring the symbol of the connective. Thus for example we shall write $x_1x_2 \dots x_n$ instead of $(\dots(x_1 \cdot x_2)\dots) \cdot x_n$. The symbol $i(\alpha)$ denotes the initial variable of the formula $\alpha \in F$ and the symbol $n(\alpha)$ denotes the set of variables occurring in α at non-initial places. A formula α is regular iff it is left associated (i.e. $\alpha = x_{f_1} \dots x_{f_n}$ where all x_{f_j} are variables) and $i(\alpha) \notin n(\alpha)$. The symbol R denotes the set of all regular formulas. First let us note two simple lemmas:

Lemma 1. *If v is a valuation of variables in the matrix $\mathbb{M}$ then the following conditions are satisfied:*

(i) if $\alpha \in F - R$ then $v(\alpha) = 2$,
(ii) if $v(\alpha) = 0$ then $v(i(\alpha)) = 0$,

(iii) if $\alpha \in R$ then $v(\alpha) = 0$ iff $v(i(\alpha)) = 0$ and $vn(\alpha) \subseteq \{1\}$.

LEMMA 2. *If $\alpha \in R$, $X \subseteq F$ and $n(\alpha) \cap X = \emptyset$ then the following conditions are equivalent:*

(i) $\alpha \in C_{\mathbb{M}}(X)$,
(ii) for some $\beta \in X \cap R$, $i(\beta) = i(\alpha)$ and $n(\beta) \subseteq n(\alpha)$.

A sequent is a pair $\langle X, \alpha \rangle$ where $\alpha \in F$, $X \subseteq F$ and $|X| < \aleph_0$. A sequent $\langle X, \alpha \rangle$ is derivable from a set of sequents B iff there exists a finite sequence of sequents:

$$\langle X_1, \alpha_1 \rangle, \ldots, \langle X_n, \alpha_n \rangle$$

such that $\langle X_n, \alpha_N \rangle = \langle X, \alpha \rangle$ and for every $i = 1, \ldots, n$ the sequent $\langle X_i, \alpha_i \rangle$ satisfies one of the following conditions (see [2]):

(i) $\alpha_i \in X_i$,
(ii) $\langle X_i, \alpha_i \rangle$ results from a sequent of B by means of a substitution,
(iii) $\langle X_i, \alpha_i \rangle$ results from some previous sequents by means of one of the following rules:
$\langle Y, \beta \rangle \Longrightarrow \langle Y \cup \{\gamma\}, \beta, \rangle$,
$\langle Y, \beta \rangle, \langle Z \cup \{\beta\}, \gamma \rangle \Longrightarrow \langle Y \cup Z, \gamma \rangle$.

It is easy to see that a sequent $\langle X, \alpha \rangle$ is derivable in the above sense from a set of sequents B if and only if $\alpha \in C_B(X)$ where C_B is the structural consequence operation determined by the basis B.

Now, for every $n = 1, 2, \ldots$, we define a set of sequences B_n putting $\langle X, \alpha \rangle \in B_n$ iff $\alpha \in C_{\mathbb{M}}(X)$ and every formula of the set $X \cup \{\alpha\}$ has at most n variables. Using induction on the length of a derivation one can prove the following:

LEMMA 3. *For every sequent $\langle X, \alpha \rangle$ derivable from the set B_n the following condition is satisfied:*

($*$) *If $\alpha \in R$ and every formula in $X \cap R$ has at least $n+1$ variables then there exists a formula $\beta \in X \cap R$ whose second (from the left) variable is just the same as the second variable of the formula α.*

Finally, for every $n = 3, 4, \ldots$ we define two formulas α_n and β_n putting:

$$\alpha_n = x_1 x_2 \ldots x_n, \qquad \beta_n = x_1 x_n \ldots x_2$$

Then $\alpha_n \in C_{\mathbb{M}}(\{\beta_n\})$ and the sequent $\langle\{\beta_n\}, \alpha_n\rangle$ is not derivable from B_{n-1} by virtue of Lemma 3. This proves the following:

THEOREM. *There is no finite basis of sequential rules for the consequence operation* $C_{\mathbb{M}}$.

References

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