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MATRIX REPRESENTATIONS FOR STRUCTURAL STRENGTHENINGS OF A PROPOSITIONAL LOGIC

Prior to investigating the degree of maximality of a propositional logic, as is the notion introduced by Ryszard Wójcicki [8], it is important to determine all structural consequence stronger than the logic considered. This characterization of all structural strengthenings was performed while establishing the degree of maximality for Łukasiewicz and Łukasiewicz-like logics [8], [4], [9], [5]. In these papers the authors used matrix consequences together with certain results concerning this notion and a representation theorem for Łukasiewicz algebras as given by R. S. Grigolia [2].

An attempt is made to characterize the structural strengthenings of any propositional logic. This characterization is closely similar to that already achieved for Łukasiewicz logics in [9], [5].

Let $\mathcal{S} = (S, F_1, F_2, \dots, F_n)$ be a propositional language, treated as a free algebra with an infinite set V of propositional variables as the set of generators. By h^e we denote the extension of the mapping $e : V \to S$ to the endomorphism of $\mathcal{S}$, $h^e \in Hom(\mathcal{S}, \mathcal{S})$.

A consequence operation Cn in the considered language is said to be structural ($Cn \in Struct$, cf. [3]) provided that $h^e(Cn(X)) \subseteq Cn(h^e(X))$ for any $X \subseteq S$ and any $e : V \longrightarrow S$. The set of all consequences (all structural consequences) in $\mathcal{S}$ is a complete lattice with the lattice ordering defined as follows:

$$Cn_1 \leqslant Cn_2 \equiv Cn_1(X) \subseteq Cn_2(X) \quad \text{for every } X \subseteq S.$$

As is known [7], for any family $\{Cn_t : t \in T\}$ of consequences (of structural consequences) in

$$(\inf_{t \in T} Cn_t)(X) = \bigcap \{Cn_t(X); t \in T\} \quad \text{for any } X \subseteq S.$$

A logical matrix for $\mathcal{S}$ is defined as a pair $M = (\mathcal{A}, I)$, where $\mathcal{A} = (A, f_1, \ldots, f_n)$ is an algebra similar to $\mathcal{S}$ and $I \subseteq A$ is the set of all designated elements of M. We say that $N = (\mathcal{B}, J)$ is a submatrix of a matrix $M = (\mathcal{A}, I)$ if, and only if, $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and $J = I \cap A$.

Given an indexed family $\{(\mathcal{A}_t, I_t)\}_{t \in T}$ of similar matrices we can form the direct product of matrices which will be considered as the pair $(P_{t\in T}\mathcal{A}_t, P_{t\in T} I_t)$, where $P_{t\in T}\mathcal{A}_t$ denotes the product of algebras.

Let K be a class of logical matrices for $\mathcal{S}$. By $SP(K)$ we denote the least class of matrices containing K and closed under the operations of forming direct products and submatrices.

If $M = (\mathcal{A}, I)$ is a matrix for $\mathcal{S}$, then the consequence determined by this matrix, denoted as Cn_M, is as follows:

$$\alpha \in Cn_M(X) \equiv \text{ for every } h \in Hom(\mathcal{S}, \mathcal{A})\ [h(X) \subseteq I \Rightarrow h(\alpha) \in I].$$

For any M, the operation Cn_M is structural. Given a class K of matrices for $\mathcal{S}$ we can define the structural consequence Cn_K by putting $Cn_K = inf_{M\in K} Cn_M$. If K is empty, then the consequence Cn_K is inconsistent, that is $Cn_K(X) = S$ for all $X \subseteq S$. We say that a class K of matrices is adequate for a consequence Cn iff $Cn = Cn_K$ (c.f. [7]). As is known [7], for any structural consequence there exists an adequate family of matrices. Thus any $Cn \in Struct$ is uniquely determined by the class of all Cn-matrices, i.e. by the class $Matr(Cn) = \{M \in Matrix; Cn \leqslant Cn_M\}$.

The following theorem states that any structural strengthening of a structural consequence can be represented by some family of matrices belonging to $SP(K)$, where K is any class of matrices adequate for the initial logic.

THEOREM. *Let K be a class of matrices for $\mathcal{S}$. Then $Cn_K \leqslant Cn \in Struct \Rightarrow Cn = Cn_L$ for some $L \subseteq SP(K)$.*

PROOF. Let $Cn_K \leqslant Cn \in Struct$ and put $L = \{M \in SP(K); Cn \leqslant Cn_M\}$. Clearly $Cn \leqslant Cn_L$, hence what we have to prove is that

$$\bigcap\{Cn_M(X); M \in SP(K) \wedge Cn \leqslant Cn_M\} \subseteq Cn(X) \text{ for any set } X \subseteq S.$$

If X is Cn-inconsistent, that is if $Cn(X) = S$, then this inclusion is obvious. In order to prove the theorem it suffices to show that for any Cn-consistent set X of formulae there exists a matrix $N \in SP(K)$ such that:

(1) $Cn \leqslant Cn_N$,
(2) $Cn(X) = Cn_N(X)$.

Let X be a certain Cn-consistent set of formulae ($Cn(X) \neq S$) and put $T = S \backslash Cn(X)$, obviously the set T is not empty. Since $Cn_K \leqslant Cn$, the identity $Cn_K(Cn(X)) = Cn(X)$ holds and therefore, using the axiom of choice and the definition of the matrix consequence, for any formula $\alpha \notin Cn(X)$ we can choose a matrix $M_\alpha = (\mathcal{A}_\alpha, I_\alpha) \in K$ and a homomorphism $h_\alpha \in Hom(\mathcal{S}, \mathcal{A}_\alpha)$ such that

(3) $h_\alpha(Cn(X)) \subseteq I_\alpha$ and $h_\alpha(\alpha) \notin I_\alpha$.

(The use of the axiom of choice is not essential but it allows the proof to be simplified).

Let us take into consideration the product $P_{\alpha \in T} M_\alpha = (P_{\alpha \in T}\mathcal{A}_\alpha, P_{\alpha \in T} I_\alpha) \in SP(K)$ and define a mapping $h : S \to P_{\alpha \in T} A_\alpha$ as follows:

(4) $h(\Phi) = \{h_\alpha(\Phi)\}_{\alpha \in T}$ for any formula $\Phi \in S$.

As is known, $h \in Hom(\mathcal{S}, P_{\alpha \in T}\mathcal{A}_\alpha)$. Therefore, the image $h(S)$ is closed under all operations in the product, that is h determines a subalgebra of $P_{\alpha \in T} A_\alpha$. This subalgebra will be denoted by $h(\mathcal{S})$.

We define the matrix N by putting $N = (h(\mathcal{S}), h(S) \cap P_{\alpha \in T} I_\alpha)$. It is easy to see that N is a submatrix of the product $P_{\alpha \in T} M_\alpha$ and consequently N belongs to $SP(K)$, i.e. the construction of the matrix N has been completed.

Proof of (1). Suppose $\Phi \in Cn(Y)$ and let $h_1 \in Hom(\mathcal{S}, h(\mathcal{S}))$ be such a homomorphism that $h_1(Y) \subseteq P_{\alpha \in T} I_\alpha$; we have to prove that $h_1(\Phi) \in P_{\alpha \in T} I_\alpha$.

For any propositional variable $\gamma \in V$, the set $h^{-1}\{h_1(\gamma)\}$ is not empty. By the axiom of choice, there exists a mapping $e : V \to S$ such that $e(\gamma) \in h^{-1}\{h_1(\gamma)\}$ for any $\gamma \in V$ (in the case where S is countable such a mapping can be defined effectively). Hence $h(e(\gamma)) = h_1(\gamma)$ for $\gamma \in V$. But the algebra $\mathcal{S}$ is freely generated by the set V; therefore $h \circ h^e = h_1$, where h^e is the endomorphism of $\mathcal{S}$ determined by the mapping e.

This shows that $h(h^e(Y)) = h_1(Y) \subseteq P_{\alpha \in T} I_\alpha$ and hence $h_\alpha(h^e(Y)) \subseteq I_\alpha$ for any $\alpha \in T$. By (3), $h^e(Y) \subseteq Cn(X)$ and, since $\Phi \in Cn(Y)$, it follows from the structurality of Cn that $h^e(\Phi) \in Cn(h^e(Y)) \subseteq Cn(X)$. Using (3) once again, we obtain $h_\alpha(h^e(\Phi)) \in I_\alpha$ for any $\alpha \in T$ and consequently

$h(h^e(\Phi)) \in P_{\alpha\in T} I_\alpha$ (see (4)). Hence $h_1(\Phi) = h(h^e(\Phi)) \in h(S) \cap P_{\alpha\in T} I_\alpha$ and this completes the proof of (1).

Proof of (2). The inclusion ($\subseteq$) follows immediately from (1). To prove the inclusion ($\subseteq$) it sufficies to consider the homomorphism $h \in Hom(\mathcal{S}, h(\mathcal{S}))$. By (3), $h(X) \subseteq h(S) \cap P_{\alpha\in T} I_\alpha$ and $h(\Phi) \notin P_{\alpha\in T} I_\alpha$ for any $\Phi \in T = S \backslash Cn(X)$. Consequently $\Phi \notin Cn_N(X)$ for any $\Phi \notin Cn(X)$, which completes the proof of (2) and thus of the whole. □

A similarity may be observed between this characterization of structural strengthenings of a sentential logic and certain representation theorems of Universal Algebra, as for example Birkhoff's theorem [1]. Despite this apparent similarity the theorem presented is not algebraic in characterize the class of all Cn_K-matrices (in general $Matr(Cn_K) \neq SP(K)$), but only states that from the point of view of logic the class $SP(K)$ is sufficiently representative.

The question arises whether a representation of structural strengthenings of a propositional logic better than the one presented here may be found. It may be possible for a certain family of logics that the class $SP(K)$ in the above theorem can be replaced by $P(S(K))$ (as in the case of Łukasiewicz logics [9]) or even by $S(K)$; obviously $S(K) \subseteq P(S(K)) \subseteq SP(K)$. From the results given in [6] it follows that, in general, the replacement of $SP(K)$ by $P(S(K))$ is impossible.

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