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'LARGE' MATRICES WHICH INDUCE FINITE CONSEQUENCE OPERATIONS

By a *propositional language* we shall mean an absolutely free algebra

$$\underline{L} = \langle L; F_1, \ldots, F_n \rangle, \ n \leqslant \omega,$$

generated by an *infinite* set $V(L)$ of absolutely free generators and equipped with at most countably many operations F_n, $n \leqslant \omega$, all of finite arity.

$P(L)$ is the power set of L. A function $C : P(L) \to P(L)$ is a *consequence operation* in $\underline{L}$ iff for each $X, Y \subseteq L$ the following conditions are satisfied:

$C1$. $X \subseteq C(X) = C(C(X))$
$C2$. if $X \subseteq Y$ then $C(X) \subseteq C(Y)$.

C is *finite* iff for each $X \subseteq L$
$C(X) = \bigcup\{C(Y) : Y \subseteq X \& Y$ is finite$\}$.

By a matrix for a propositional language $\underline{L}$ we shall mean a couple

$$\mathcal{M} = (\underline{A}, D),$$

where $\underline{A}$ is an algebra similar to $\underline{L}$ and D is a subset of A (A – the carrier of $\underline{A}$). Every matrix $\mathcal{M}$ for $\underline{L}$ induces the consequence operation $Cn_{\mathcal{M}}$ on L, where

$$\begin{array}{lll} \alpha \in Cn_{\mathcal{M}}(X) & \text{iff for all} & h \in Hom(\underline{L}, \underline{A}), \\ h(\alpha) \in D & \text{whenever} & h(X) \subseteq D. \end{array}$$

$Hom(\underline{L}, \underline{A})$ is the set of all valuations of $\underline{L}$ in $\mathcal{M}$, i.e. homomorphisms from $\underline{L}$ into $\underline{A}$.

Let I be a nonempty set and $\mathcal{F}$ – an ultrafilter over I. In the usual model – theoretic way we define the ultraproduct $\mathcal{M} = \prod_{i \in I} {}_{\mathcal{F}} \mathcal{M}_i$ of matrices $\mathcal{M}_i = (\underline{A}_i, D_i)$, $i \in I$, modulo $\mathcal{F}$.

A filter $\mathcal{F}$ over a non empty set I is said to be *countably incomplete* iff $\mathcal{F}$ is not closed under countable intersections.

Our purpose is to prove the following theorem.

THEOREM A. *Let $\mathcal{M}_i = (A_i, D_i)$, $i \in I$, be a non empty family of matrices for a countable propositional language $\underline{L}$. Let be a countably incomplete ultrafilter over a set I. Then the consequence operation induced by the matrix $\mathcal{M} = \prod_{i\in I}{}_{\mathcal{F}}\mathcal{M}_i$ in $\underline{L}$ is finite.*

PROOF (CF. THEOREM 6.1.1 IN [1], P. 305). Let $\beta \in L$, $X = \{\gamma_n : n \in N\} \subseteq L$ be an infinite set of propositional formulas such that for each finite $Y \subset X$, $\beta \notin Cn_{\mathcal{M}}(Y)$. Hence for each $Y \subset X$, Y-*finite*, there is a valuation $h^Y = \langle h_i^Y : i \in I\rangle$ of $\underline{L}$ in $\prod_{i\in I} \mathcal{M}_i$ such that

$$S_Y =_{df} \bigcap_{\gamma\in Y} \{i : h_i^Y(\gamma) \in D_i\} \cap \{i : h_i^Y(\beta) \notin D_i\} \in \mathcal{F}$$

Let $Y_n = \{\gamma_1, \ldots, \gamma_n\}$, $n \geqslant 1$. As $\mathcal{F}$ is countably incomplete, we find a descending chain $I = I_0 \supset I_1 \supset I_2 \supset \ldots$ such that each $I_n \in \mathcal{F}$ and $\bigcap_{n<\omega} I_n = \emptyset$. Let $T_0 = I$ and for each positive $n < \omega$ let $T_n = I_n \cap S_{Y_n}$. Clearly $T_n \in \mathcal{F}$, $\bigcap_{n<\omega} T_n = \emptyset$, $T_n \supseteq T_{n+1}$. It follows that for each $i \in I$ there is a greatest $n(i) < \omega$ such that $i \in T_{n(i)}$.

Define a valuation $h = \langle h_i : i \in I\rangle$ in $\prod_{i\in I} \mathcal{M}_i$ as follows:

(i) if $n(i) = 0$, then h_i is an arbitrary valuation of $\underline{L}$ in $\mathcal{M}_i$

(ii) if $n(i) > 0$, then h_i is a valuation of $\underline{L}$ in $\mathcal{M}_i$ such that

$$(*) \qquad h_i(Y_{n(i)}) \subseteq D_i \ \text{ and } \ h_i(B) \notin D_i.$$

Notice that if $n(i) > 0$ then the valuation $h_i^{Y_{n(i)}}$ fulfils $(*)$. Indeed, if $n(i) > 0$ then

$$i \in S_{Y_{n(i)}} = \bigcap_{\gamma\in Y_{n(i)}} \{j : h_j^{Y_{n(i)}}(\gamma) \in D_j\} \cap \{j : h_j^{Y_{n(i)}}(B) \notin D_j\}.$$

Thus a valuation h with properties (i) - (ii) exists.

We have:

for each $n \geqslant 1$ $T_n \subseteq \{i : h_i(\gamma_n) \in D_i\}$.

Indeed, if $n > 0$ and $i \in T_n$ then $n \leqslant n(i)$. Hence $\gamma_n \in Y_{n(i)}$ and, consequently, by $(*)$ $h_i(\gamma_n) \in D_i$ in $\mathcal{M}_i$. Thus we have proved: for every $\gamma \in X$ $\{i : h_i(\gamma) \in D_i\} \in \mathcal{F}$.

Now notice that if $n(i) > 0$ then $h_i(B) \notin D_i$.
But $T_1 = \{i : n(i) > 0\}$.
Hence $\{i : h_i(B) \notin D_i\} \in \mathcal{F}$, i.e., $\{i : h_i(B) \in D_i\} \notin \mathcal{F}$.
Consequently $h_{\mathcal{F}}(X) \subseteq \prod_{i \in I} {}_{\mathcal{F}} D_i$ and $h_{\mathcal{F}}(B) \notin \prod_{i \in I} {}_{\mathcal{F}} D_i$. Therefore $B \notin Cn(X)$.

Corollary. *Let $\{\mathcal{M}_n : n \in \omega\}$ be a countable family of matrices. Let $\mathcal{F}$ be a non-principial ultrafilter over ω. Then the consequence operation induced by $\prod_{n \in \omega} {}_{\mathcal{F}} \mathcal{M}_n$ in a countable propositional language $\underline{L}$ is finite.*

For an uncountable $\underline{L}$ (i.e. when $V(\underline{L})$ has the cardinality $> \aleph_0$) we state the following theorem.

Theorem B. *Let $\underline{L}$ be a propositional language of power $< \mathfrak{m}$. Let $\mathcal{M}_i$, $i \in I$, be a nonempty family of matrices for $\underline{L}$. Let $\mathcal{F}$ be a countably incomplete $\mathfrak{m}$-good ultrafilter over a set I. Then the consequence operation induced by the matrix $\mathcal{M} = \prod_{i \in I} {}_{\mathcal{F}} \mathcal{M}_i$ in $\underline{L}$ is finite.*

Proof resembles the proof of Theorem A (cf. Theorem 6.1.8, [1], p. 313). For a definition of $\mathfrak{m}$-good ultrafilters consult [1].

References

[1] C. C. Chang and H. J. Keisler, **Model theory**, North-Holland American Elsevier, Amsterdam, New York, 1973.

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