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A CHARACTERIZATION OF $Matr(C)$

This is an abstract of the paper "Reduced products of logical matrices" submitted to Studia Logica.

Propositional language are defined as in the note [2]. If $\underline{L}$ and $\underline{L}'$ are two propositional languages and $\underline{L}$ is a subalgebra of $\underline{L}'$ then $\underline{L}$ will be called a sublanguage of $\underline{L}'$ and $\underline{L}'$ an extension of $\underline{L}$, symbolically $\underline{L} \subseteq \underline{L}'$. Notice that a subalgebra of $\underline{L}'$ need not be a sublanguage of $\underline{L}'$.

Let C be a consequence operation on a language $\underline{L}$. By the cardinality of C, $card(C)$, we shall mean the least cardinal n such that for all $X \subseteq L$

$$C(X) = \bigcup\{C(Y)|Y \subseteq X \ \& \ \bar{\bar{Y}} < \mathfrak{n}\}$$

C is finite iff $card(C) < \aleph_0$.

Given two consequence operations C_1, C_2 on $\underline{L}$ we write $C_1 \leqslant C_2$ iff $C_1(X) \subseteq C_2(X)$, all $X \subseteq L$.

If $\mathcal{M} = (\underline{A}, D)$ is a matrix for $\underline{L}$ then $Cn_{\mathcal{M},\underline{L}}$ is the consequence operation on $\underline{L}$ induced by $\mathcal{M}$. If $\mathbb{K}$ is a class of matrices for $\underline{L}$ then

$$Cn_{\mathbb{K},\underline{L}} =_{df} \inf_{\mathcal{M}\in\mathbb{M}} Cn_{\mathcal{M},\underline{L}}$$

where the infimum is taken with respect to $\leqslant$.

If C is consequence operation on $\underline{L}$ then we define $Matr(C)$ to be the class of all matrices $\mathcal{M}$ for $\underline{L}$ such that $C \leqslant Cn_{\mathcal{M},\underline{L}}$.

A consequence operation C on $\underline{L}$ is *structural* provided that $\varepsilon(C(X)) \subseteq C(\varepsilon(X))$, all $X \subseteq L$, all endomorphisms ε of $\underline{L}$.

By an $\mathfrak{m}$-filter over a set I we shall mean a filter $\mathcal{F}$ closed with respect to intersections of fewer than $\mathfrak{m}$ members of $\mathcal{F}$. Every improper filter $\mathcal{F}$ is an $\mathfrak{m}$-filter.

Let $\mathbb{K}$ be a class of matrices for $\underline{L}$. We use the following terminology:

$S(\mathbb{K})$ – the class of all isomorphic images of submatrices of members of $\mathbb{K}$,

$P(\mathbb{K})$ – the class of all isomorphic images of direct products of arbitrary systems (possibly empty) of members of $\mathbb{K}$,

$P_{\mathfrak{m}-r}(\mathbb{K})$ – the class of all isomorphic images of $\gtrdot$-reduced products (i.e. direct products modulo $\mathfrak{m}$-filters) of arbitrary systems of members of $\mathbb{K}$,

$H_s(\mathbb{K})$ – the class of all strong homomorphic images of members of K,

$\overleftarrow{H_s}(\mathbb{K})$ – the class of all strong homomorphic counter – images of members of $\mathbb{K}$. Hence $\mathcal{M} \in \overleftarrow{H_s}(\mathbb{K})$ iff there exists a strong homomorphisms from $\mathcal{M}$ onto a matrix in $\mathbb{K}$. Clearly $\mathbb{K} \subseteq H_s(\mathbb{K})$ and $\mathbb{K} \subseteq \overleftarrow{H_s}(\mathbb{K})$.

If $\mathfrak{m} = \aleph_0$ then we write

$P_r(\mathbb{K})$ instead of $P_{\aleph_0-r}(\mathbb{K})$.

THEOREM 1. *Let C be a structural consequence operation on a propositional language $\underline{L}$. Let $\{\mathcal{M}_i\}_{i\in I}$ be a family of matrices for $\underline{L}$ such that $C \leqslant Cn_{\mathcal{M}_i,\underline{L}}$, all $i \in I$. Let $\mathfrak{m}$ be an infinite cardinal $\geqslant card(C)$ and let be an $\mathfrak{m}$-filter over I.*

Form the $\mathfrak{m}$-reduced product $\mathcal{M} = \prod_{i\in I} {}_{\mathcal{F}}\mathcal{M}_i$. Then $C \leqslant Cn_{\mathcal{M},\underline{L}}$.

THEOREM 2. *Let $\mathbb{K}$ be a class of matrices for a propositional language $\underline{L}$. Let $\mathfrak{m}$ be a regular infinite cardinal such that $card(Cn_{\mathbb{K},\underline{L}}) \leqslant \mathfrak{m} \leqslant \bar{\bar{L}}^+$. Then*

$$Matr(Cn_{\mathbb{K},\underline{L}}) = \overleftarrow{H_s} H_s S P_{\mathfrak{m}-r}(\mathbb{K}).$$

For a given structural C in $\underline{L}$ let

$Th(C) =_{df} \{X | X \subseteq L \ \&\ C(X) = X\}$ and

$\mathbb{L}_C =_{df} \{\underline{L}_X | \underline{L}_X = (\underline{L}, X) \ \&\ X \in Th(C)\}$.

$\mathbb{L}_C$ is called the *Lindenbaum bundle* of C. As known $C = Cn_{\mathbb{L}_C,\underline{L}}$ [1]. Let $\mathbb{L}_C^* = \{\underline{L}_{X/\theta_X} | X \in Th(C)\}$, where θ_X is the greatest congruence is the matrix $\underline{L}_X$ (see [1]).

THEOREM 3. *Let C be a structural consequence operation on a propositional language $\underline{L}$. Let $\mathfrak{m}$ be an infinite regular cardinal such that $card(C) \leqslant \mathfrak{m} \leqslant \bar{\bar{L}}^+$. Then*

$$Matr(C) = \overleftarrow{H_s}\ S\ P_{\mathfrak{m}-r}(\mathbb{L}_C^*).$$

THEOREM 4. *Let $\mathbb{K}$ be a class of matrices for a propositional language $\underline{L}$.*

The following are equivalent:

(a) $Matr(Cn_{\mathbb{K},\underline{L}}) = \overleftarrow{H_s} H_s SP(\mathbb{K})$

(b) *For every extension* $\underline{L}'$ *of* $\underline{L}$

$$Matr(Cn_{\mathbb{K},\underline{L}}) \subseteq Matr(Cn_{\mathbb{K},\underline{L}'}).$$

THEOREM 5. *Let* $\mathbb{K}$ *be a finite family of finite matrices for a propositional language* $\underline{L}$. *Then*

$$Matr(Cn_{\mathbb{K},\underline{L}}) = \overleftarrow{H_s} H_s SP(\mathbb{K}).$$

For a given class $\mathbb{K}$ *of matrices for* $\underline{L}$ ($\underline{L}$*-fixed) define*

$$\mathcal{M} \sim \mathcal{N} \textit{ iff } Cn_{\mathcal{M},\underline{L}} = Cn_{\mathcal{N},\underline{L}}$$

$$\mathcal{M} \precsim \mathcal{N} \textit{ iff } Cn_{\mathcal{M},\underline{L}} \leqslant Cn_{\mathcal{N},\underline{L}}$$

$(\mathcal{M}, \mathcal{N} \in \mathbb{K})$. *Then* $\mathbb{K}/_\sim$ *is a set and* $\mathbb{K}/_\sim$ *is partially ordered by* $\precsim$.

THEOREM 6. *Let* $\mathbb{K}$ *be a finite class of matrices for a propositional language* $\underline{L}$. *Then the partially ordered sets* $\langle Matr(Cn_{\mathbb{K},\underline{L}})/_\sim, \precsim\rangle$ *and* $\langle SP(\mathbb{K})/_\sim, \precsim\rangle$ *are isomorphic.*

References

[1] R. Wójcicki, *Matrix approach in methodology of sentential calculi*, **Studia Logica**, Vol. XXXII 1973.

[2] J. Czelakowski, *'Large' matrices which induce finite consequence operations*, this **Bulletin**, pp. 79–82.

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