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AN EXAMPLE OF STRONGLY FINITE CONSEQUENCE OPERATION WITH $2^{\aleph_0}$ STANDARD STRENGTHENINGS

The paper gives the affirmative answer to the following question: are there strongly finite logics with the degree of maximality greater than $\aleph_0$. The question was posed by M. Tokarz in [4].

Let $\mathcal{A} = ((\{0,1,2,3,4,5,6\}, \circ), \{1,2,3\})$ be a matrix whose the binary operation $\circ$ is given as follows:

$x \circ y$	0	1	2	3	4	5	8
0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0
2	0	2	2	0	0	0	0
3	0	0	0	5	3	3	0
4	0	0	0	3	4	3	0
5	0	0	0	3	3	5	0
6	0	0	6	0	0	0	1

Notice that the algebra of the submatrix of $\mathcal{A}$ generated by the set $\{0,1,2\}$ is the Murskiǐ's algebra (see [3]), while the submatrix of $\mathcal{A}$ generated by $\{3,4,5\}$ is isomorphic with the matrix considered by A. Wroński in [5].

It is easy establish the following

Lemma 1. *Let α be a formula and v a valuation in $\mathcal{A}$ such that $v(Var) \subseteq \{2,6\}$. Then, $v(l(\alpha)) = 6$ only if $v(\alpha) \in \{0,1,6\}$, where $l(\alpha)$ denotes the variable occurring in α first from the left.*

The following formulae play a central role in our considerations: $\beta_{2n+1} = \alpha_{2n+1}(x_0/x_0x_0,\ x_1/x_1x_1, \ldots, x_{2n+1}/x_{2n+1}x_{2n+1})$, $n \geqslant 1$ where all α_{2n+1} $(n \geqslant 1)$ are defined in §2 of [1].

LEMMA 2. *For any k, n the following conditions hold:*

(i) $Cn_{\mathcal{A}}(\beta_{2k+1}) \neq L$

(ii) if $k \neq n$ *then* $e\beta_{2k+1} \notin Cn_{\mathcal{A}}(\beta_{2n+1})$ *for all* $e \in Hom(\underline{L}, \underline{L})$.

PROOF. (i) Take a valuation v in $\mathcal{A}$ such that $v(x_{2k+2}) = 1$ and $v(x) = 2$ otherwise, and observe that $v(\beta_{2k+1}) = 2$ and $v(x_{2k+2}x_{2k+2}) = 0$.

(ii) CASE 1: $k > n$. Assume $e\beta_{2k+1} \in Cn_{\mathcal{A}}\ (\beta_{2n+1})$ for some $e \in Hom(\underline{L}, \underline{L})$. Hence, we have $Var(e\beta_{2k+1}) \subseteq Var(\beta_{2n+1})$. Indeed, if it were not true, then there would be x_j such that $x_j \in Var(e\beta_{2k+1}) \backslash Var(\beta_{2n+1})$, and then taking a valuation v in $\mathcal{A}$ defined as follows: $v(x) = 0$ when $x = x_j$, and $v(x) = 2$ otherwise, we would have $v(e\beta_{2k+1}) = 0$ and $v(\beta_{2n+1}) = 2$, but this contradicts our assumption. Thus $ex_i \in L^{(2n+2)}$ for all $x_i \in Var(\beta_{2k+1})$, where $\underline{L}^{(2n+2)}$ is a sublanguage of $\underline{L}$ generated by $\{x_i; i < 2n+2\}$. Then, since $k > n$ we have $l(ex_i) = l(ex_j)$ for some $x_i, x_j \in Var(\beta_{2k+1})$. Now, take a valuation v in $\mathcal{A}$ such that $v(x) = 6$, when $x = l(ex_i)$, and $v(x) = 2$, otherwise. We have for it by Lemma 1, $v(ex_i)$, $v(ex_j) \in \{0, 1, 6\}$, furthermore $((ex_iex_i)(ex_jex_j)) = 0$, so consequently $v(e\beta_{2k+1}) = 0$. On the other hand, by Lemma 2.2 and 2.3 in [1], we obtain $v(\beta_{2n+1}) \in \{1, 2\}$, but this and the initial assumption yield a contradiction. Thus the proof of CASE 1 is complete.

CASE 2: $k < n$ (cf. the proof of Lemma 1.1 of [5]). Let e be a substitution, i.e. $e \in Hom(\underline{L}, \underline{L})$. Since $k < n$ then there must exist $x_j \in Var(\beta_{2n+1})$ such that $x_j \notin Var(ex_i)$ or $\{x_j\} \subsetneq Var(ex_i)$, for all $x_i \in Var(\beta_{2k+1})$. Take a valuation v in $\mathcal{A}$ defined as follows: $v(x) = 4$, when $x = x_j$, and $v(x) = 5$, otherwise, and notice that we have for it $v(ex_i) \in \{3, 5\}$, for all $x_i \in Var(\beta_{2k+1})$, what implies $v(e\beta_{2k+1}) = 5$. To complete the proof we must show now $(*) v(\beta_{2k+1}) = 3$.

SUBCASE 2a: j is odd. Denote by δ_i^{*k} $(i < k)$ the formula $\delta_i^k(x_i/x_ix_i, x_{i+1}/x_{i+1}x_{i+1}, \ldots, x_k/x_kx_k)$, where δ_i^k is defined in §2 of [1]. Since $(2n+1) - j$ is even we have $v(\delta_j^{*2n+1}) = 5$. Also $v(\delta_k^{*2n+1}) = 5$ $(j < k < 2n+1)$ because x_j does not occur in δ_k^{*2n+1}. In every δ_k^{2n+1} $(0 \leqslant k < j)$ x_j has exactly one occurrence, so, $v(\delta_k^{*2n+1}) = 3$ whenever $0 \leqslant k < j$. Thus, since $card\{k; k < j\}$ is odd, we get $v(\delta_0^{*2n+1} \ldots \delta_{2n}^{*2n+1}) = 3$ and, consequently, $v(\beta_{2n+1}) = 3$.

SUBCASE 2b: j is even. The proof of $(*)$ goes here similarly to the previous one.

In this way we obtain $v(e\beta_{2k+1}) = 5$ and $v(\beta_{2n+1}) = 3$ which give us

$e\beta_{2k+1} \notin Cn_{\mathcal{A}}(\beta_{2n+1})$, and so the proof is complete. QED

THEOREM 2. *card*$\{Cn; Cn$ *is a standard consequence operation on* $\underline{L}$ *such that* $Cn_{\mathcal{A}}\} \leqslant Cn = 2^{\aleph_0}$.

PROOF. For a non-empty subset A of $\{\beta_{2n+1}; n \geqslant 1\}$, let Cn_A be a consequence operation on $\underline{L}$ determined by a set of rules obtained by adding to some fixed standard basis of $Cn_{\mathcal{A}}$ all rules of the form $\{(e\beta_{2n+1}, ex_{2n+2}); e \in Hom(\underline{L}, \underline{L})\}$ for $\beta_{2n+1} \in A$. From the definition of Cn_A and via Łoś-Suszko's theorem (see [2]) we know that all Cn_A are standard strengthening of $Cn_{\mathcal{A}}$. Therefore, to complete the proof it is enough to show that $Cn_A \neq Cn_B$, whenever $A \neq B$. Assume that $A \backslash B \neq \emptyset$ and take any $\beta_{2n+1} \in A \backslash B$. Using Lemma 1 we get $Cn_B(\beta_{2n+1}) \subseteq Cn_{\mathcal{A}}(\beta_{2n+1}) \neq L$, but $Cn_A(\beta_{2n+1}) = L$ so, indeed, $Cn_A \neq Cn_B$. QED

References

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