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ω-SATURATED MATRICES

The paper is a continuation of the investigations of [3]. We assume that the reader is familiar with the notion of an ω-saturated model and the notation applied in the theory of models. We refer to [2] for all model-theoretic terms which appear in this note.

By a propositional language we shall mean a *countable* absolutely free algebra

$$\underline{L} = \langle L; F_1, \ldots, F_n \rangle,\ n \leqslant \omega$$

with infinitely many absolutely free generators $V(L) = \{p_n : n \in \omega\}$. Every operation F_k in $\underline{L}$ has a finite arity. By an elementary matrix for a propositional language $\underline{L}$ we shall mean a pair $\mathfrak{M} = (\underline{A}, D)$, where $\underline{A}$ is an algebra similar to $\underline{L}$ and D is a subset of A (A – the carrier of $\underline{A}$). If $\mathfrak{M} = (\underline{A}, D)$ is a matrix for $\underline{L}$, then $Cn_{\mathfrak{M}}$ is the consequence operation in $\underline{L}$ induced by $\mathfrak{M}$:

$$\alpha \in Cn_{\mathfrak{M}}(X) \text{ iff } \forall h \in Hom(\underline{L}, \underline{A})$$

$$[h(X) \subseteq D \Rightarrow h(\alpha) \in D]\ (\alpha \in L, X \subseteq L).$$

Let $\underline{L}$ be a propositional language. Let $\underline{L}^*$ be the first-order language with equality of the class of all matrices for $\underline{L}$. $\underline{L}^*$ has one unary predicate letter, say D. We assume that the variables of $\underline{L}^*$ are identified with the elements of $V(\underline{L})$, the set of free generators of $\underline{L}$. Thus the algebra of terms of $\underline{L}^*$ is identical with the propositional language $\underline{L}$. Consequently, the models for $\underline{L}^*$ are exactly matrices for $\underline{L}$ (cf. [1]).

A matrix $\mathfrak{M} = (\underline{A}, D)$ is ω-saturated iff it is ω-saturated in the usual sense [2, p. 96] as a model for $\underline{L}^*$.

Our aim is to prove the following theorem:

THEOREM. *Let $\mathfrak{M} = (\underline{A}, D)$ be an elementary matrix for a countable propositional language $\underline{L}$. Assume that $\mathfrak{M}$ is ω-saturated. Then the consequence operation $Cn_{\mathfrak{M}}$ induced by $\mathfrak{M}$ in $\underline{L}$ is finitistic.*

PROOF. Let $X \subseteq L$, $\alpha \in L$, and assume that for every $Y \subset X$, if $\bar{\bar{Y}} < \omega$ then $\alpha \notin Cn_{\mathfrak{M}}(Y)$. It means that every *finite* subset of the following set of open formulas from L^*

$$\Sigma_0 =_{df} \{D(\gamma) : \gamma \in X\} \cup \{\neg D(\alpha)\}$$

is realized in $\mathfrak{M}$. Assume that $\Sigma_0 = \Sigma_0(x_1 x_2 \ldots x_n \ldots)$. Let Σ be the closure Σ_0 under finite conjunctions. Define:

$$\Sigma_n = \{\sigma \in \Sigma | \sigma = \sigma(x_1 \ldots x_n)\}, \text{ all } n \in \omega$$

and

$$\Sigma^{(k)} = \bigcup_{n \geqslant k+1} \{\exists x_{k+1} \ldots \exists x_n \sigma | \sigma \in \Sigma_n\}, \text{ all } k \in \omega.$$

Notice that $\Sigma^{(k)} = \Sigma^{(k)}(x_1 \ldots x_k)$. In particular, $\Sigma^{(0)}$ is the set of existential closures formulas from Σ.

To the set of symbols of the language L^* add a countable number of constant symbols $\{C_n : n \in N\}$. For each $k \in N$ and each $1 \leqslant m \leqslant k$, let

$$\Sigma^{(k)}_{c_1 \ldots c_m} =_{df} \{\sigma(x_1/c_1 \ldots x_m/c_m) | \sigma \in \Sigma^{(k)}\}.$$

Thus $\Sigma^{(k)}_{c_1 \ldots c_m}$ is a set of formulas of $L^* \cup \{c_1, \ldots, c_m\}$. Moreover $\Sigma^{(k)}_{c_1 \ldots c_m} = \Sigma^{(k)}_{c_1 \ldots c_m}(x_{m+1} \ldots x_k)$.

Consider $\Sigma^{(1)}$. Σ is consistent with $Th(\mathfrak{M})$ and it is closed under conjunctions. Consequently, $\Sigma^{(1)}$ is consistent with $Th(\mathfrak{M})$. Since $\mathfrak{M}$ is ω-saturated, there is $a_1 \in A$ such that $\mathfrak{M} \models \Sigma^{(1)}[x_1/a_1]$. Take $L^* \cup \{c_{a_1}\}$. Then $\Sigma^{(1)}_{c_{a_1}} \subset Th(\mathfrak{M}, a_1)$. Consequently, since Σ is closed under conjunctions, $\Sigma^{(2)}_{c_{a_1}}$ is also consistent with $Th(\mathfrak{M}, a_1)$. Due to ω-saturation of $\mathfrak{M}$, there is $a_2 \in A$ such that $(\mathfrak{M}, a_1) \models \Sigma^{(2)}_{c_{a_1}}[x_2/a_2]$. Thus we inductively define a countable sequence $(a_1, a_2, \ldots, a_k, \ldots)$ of elements of A such that

$$\mathfrak{M} \models \Sigma^{(1)}[x_1/a_1]$$

and for each $k \in N$

$$(\mathfrak{M}, a_1, \ldots, a_k) \models \Sigma^{(k+1)}_{c_{a_1} \ldots c_{a_k}}[x_{k+1}/a_{k+1}].$$

Therefore for each $k \geqslant 1$

$$\Sigma^{(k)}_{c_{a_1} \dots c_{a_k}} \subset Th(\mathfrak{M}, a_1, \dots, a_k).$$

It follows that

$$\bigcup_{k \in N} \Sigma^{(k)}_{c_{a_1} \dots c_{a_k}} \subset Th(\mathfrak{M}, a_1, \dots, a_k, \dots).$$

Hence

$$\mathfrak{M} \models \Sigma(x_1 \dots x_n \dots)[x_1/a_1, \dots, x_n/a_n, \dots],$$

i.e., The set Σ is realized in $\mathfrak{M}$ by the sequence $(a_1, \dots, a_n, \dots)$. It means that

$$\alpha \notin Cn_{\mathfrak{M}}(X).$$

Q.E.D.

Remark. As a corollary we obtain Theorem A from [3]. Indeed, if $\mathfrak{F}$ is a countably incomplete ultrafilter over I, then the matrix $\mathfrak{M} = \Pi_{i \in I} \mathfrak{M}_i$ is ω_1-saturated [2, Theorem 6.1.1]. Hence $\mathfrak{M}$ is ω-saturated.

References

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