Bulletin of the **S**ection of **L**ogic
Volume 8/3 (1979), pp. 130–132

reedition 2010 [original edition, pp. 130–132]

Roman Suszko and Tamara Weinfeld

FILTERS AND NATURAL EXTENSIONS OF CLOSURE SYSTEMS

(Presented at the Logic Seminar, Wrocław, March 26, 1979)

This note calls attention to the fact that the natural extensions of standard (i.e., structural and algebraic) logics appear to be an application of the method of inductive generation of logics (in the sense of [1]). One can generalize our observations beyond the range of algebraic logics under suitable conditions on cardinals of certain involved sets.

I. Let A_0 be an algebra of a fixed similarity type with a countable stock of finitary operations. Suppose, the type of an algebra A is equal or greater than that of A_0, that is, the set $Hom(A_0, A)$ is well defined. If $\underline{C}$ is a closure system on $|A_0|$ then the class of $\underline{C}$-filters in A, $F_C(A)$, is defined as the closure on $|A|$, inductively generated from $\underline{C}$ by $Hom(A_0, A)$, i.e., the set of all $Y \subseteq A$ such that the h-counter-image $\overleftarrow{h}(Y)$ belongs to $\underline{C}$ for all $h \in Hom(A_0, A)$. In fact, $F_{\underline{C}}(A)$ is the greatest closure system $\underline{D}$ on A such that for all $X \subseteq |A_0|$ and $h \in Hom(A_0, A) : h(C(X)) \subseteq D(h(X))$; here, C and D are closure operators corresponding to $\underline{C}$ and $\underline{D}$, respectively.

II. Consider a sentential language L_0 with a countable infinite set of variables and a fixed countable set of connectives. One may be concerned with *two* kinds of extensions of L_0, both labelled as L:

(1) L is L_0 supplemented with some additional new variables, and/or
(2) L is L_0 supplemented with a countable set of new connectives.

In addition to the set $Hom(L_0, L)$ we have the set $End(L|L_0)$ defined as the family of all those maps of L into itself which preserve all connectives of L_0. In the 1-st kind case, the set $End(L|L_0)$ reduced to the set of all endomorphisms of L, $End(L)$. In the 2-nd kind case, $End(L|L_0)$ is a proper overset of $End(L)$.

If L is an extension of L_0 (of whichever kind), and $\underline{C}$ and $\underline{D}$ are closure systems on $|L_0|$ and $|L|$, respectively, then:

(a) $\underline{C}$ is structural in L_0 if $\overleftarrow{e}(X)$ is an $\underline{C}$ whenever $X \in \underline{C}$ and $e \in End(L_0)$;
(b) $\underline{D}$ is structural in $L|L_0$ if $\overleftarrow{e}(Y)$ is in $\underline{D}$ whenever $Y \in \underline{D}$ and $e \in End(L|L_0)$;
(c) $\underline{D}$ is an (conservative, no-creative) extension of $\underline{C}$ if $\underline{C}$ is the collection of all intersections $Y \cap |L_0|$ for arbitrary $Y \in \underline{D}$.

III. Let L be an extension of L_0. Suppose $\underline{C}$ is a closure system on $|L_0|$. Then, one easily proves the following.

(3) $F_{\underline{C}}(L_0) \subseteq C \cap F_{\underline{C}}(L)$, and $F_{\underline{C}}(L)$ is structural in $L|L_0$ and an extension of $F_{\underline{C}}(L_0)$.
(4) $F_{\underline{C}}(L)$ is algebraic whenever $\underline{C}$ is.
(5) If $\underline{C}$ is structural in L, the $\underline{C} = F_{\underline{C}}(L_0)$ and, hence, $F_{\underline{C}}(L)$ is an extension of $\underline{C}$; moreover, the closure operator on $|L|$ corresponding to $F_{\underline{C}}(L)$, D, is given for $b \in |L| \supseteq Y$ as follows: $b \in D(Y)$ iff there exist $a \in |L_0|$, $X \subseteq |L_0|$ and $h \in Hom(L_0, L)$ such that $h(a) = b$, $h(X) \subseteq Y$ and $a \in C(X)$; here, C is the closure operator on $|L_0|$ corresponding to $\underline{C}$.
(6) If $\underline{C}$ is standard in L_0, then $F_{\underline{C}}(L)$ is the unique algebraic and structural in $L|L_0$ closure system in L which extends $\underline{C}$; $F_{\underline{C}}(L)$ is said to be a natural extension of $\underline{C}$.

In the 1-st kind case, an alternative and somewhat intricate definition of natural extensions has been given in [2].

IV. To back up the label "natural" we add the following observation. Suppose L is a 1-st kind extension of L_0 and $\underline{D}$ is a natural extension in L of a standard $\underline{C}$ in L_0. Then, L_0 and L are of the same type and, for any algebra A of that type:

(7) $F_{\underline{C}}(A) = F_{\underline{D}}(A)$.

References

[1] D. J. Brown and R. Suszko, *Abstract Logics*, **Dissertationes Mathematicae** 102 (1973), pp. 9–41.

[2] J. Łoś and R. Suszko, *Remarks on Sentential Logics*, **Indagationes Mathematicae** 20 (1958), pp. 177–183.

Section of Logic
Institute of Philosophy and Sociology
Polish Academy of Sciences