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# SOME REMARKS ON THE LOGIC OF VAGUENESS

Our objective here is to offer a contribution to the study of the logic of vagueness. We restrict ourselves to "vagueness related to negation". This kind of vagueness happens when the law of excluded middle or the law of contradiction is not valid. We also consider only cases in which these two laws are not equivalent (when these laws are not valid but equivalent, we obtain systems related to dialectical logic like those suggested in [3]). Consequently, we have four cases of vagueness related to negation: 1) The law of excluded middle is valid but not the law of contradiction; 2) The law of contradiction is valid but not the law of excluded middle; 3) One of these laws is valid but not both; 4) Both these laws are not valid. For each case we present a logical system that formalizes the situation. We develop here only the propositional calculi; the corresponding first-order predicate calculi are easy to construct. The logical system for the first case is the system $C_1$ of da Costa [2]. In a certain sense, the logical systems for the first two cases are "duals". The semantical study of the systems presented here will be done in [1].

## 1. The system $V_0$

This is the system corresponding to the case in which the law of excluded middle and the law of contradiction are not valid, and not also equivalent. In this section as well as in the following, the terminology and notations are those of Kleene [4].

Definition.

$A^0 =_{df} \neg(A \,\&\, \neg A)$ $\qquad$ $A^+ =_{df} {}^0A \,\&\, A^0$

${}^0A =_{df} A \vee \neg A$ $\qquad$ $\neg^* A =_{df} A \supset (\neg A \,\&\, A^0)$

The postulates of $V_0$ are the following:

1. $A \supset (B \supset A)$
2. $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
3. $A, A \supset B / B$
4. $A \,\&\, B \supset A$
5. $A \,\&\, B \supset B$
6. $A \supset (B \supset A \,\&\, B)$
7. $A \supset A \vee B$
8. $B \supset A \vee B$
9. $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
10. ${}^0A \,\&\, B^0 \,\&(A \supset B) \,\&\, (A \supset \neg B) \supset \neg A$
11. $A^0 \,\&\, B^0 \supset (A \supset B)^0 \,\&\, (A \,\&\, B)^0 \,\&\, (A \vee B)^0$
12. ${}^0A \,\&\, {}^0B \supset^0 (A \supset B) \,\&\, {}^0(A \,\&\, B) \,\&\, {}^0(A \vee B)$
13. $A^0 \supset (\neg A)^0$
14. ${}^0A \supset^0 (\neg A)$
15. $\neg^*\neg^* A \supset A$

THEOREM 1.1. *The strong negation on $V_0(\neg^*)$ has all the properties of the classical negation.*

PROOF. It is enough to prove $(A \supset B)\&(A \supset \neg^* B) \supset \neg^* A$.

THEOREM 1.2. *Suppose that the formula of $\Gamma \cup \{F\}$ are formulas of $C_0$ (the classical propositional calculus), whose atomic components are $P_1, \ldots, P_n$. If $\Gamma \vdash_{C_0} F$, then $P_1^+, \ldots, P_n^+, \Gamma \vdash_{V_0} F$.*

THEOREM 1.3. *Let $A$ be a formula of $C_0$ and $A^*$ the formula obtained from $A$ by replacing $\neg$ by $\neg^*$. If $\vdash_{C_0} A$, then $\vdash_{V_0} A^*$.*

THEOREM 1.4. *If $\vdash A$ in the classical positive logic, then $\vdash A$ in $V_0$.*

PROOF. Observe that Peirce's law is a theorem of $V_0$.

THEOREM 1.5. *In $V_0$ we prove the following schemas and rules:*

| | |
|---|---|
| $A \,\&\, B^0 \,\&(A \supset B)\&(A \supset \neg B) \supset \neg A$ | *If* $\vdash A$ *then* $\vdash^0 A$ |
| $A^0 \,\&\, {}^0B, \neg B \supset \neg A \vdash A \supset B$ | $A \,\&(A \supset B)\&(A \supset \neg B \,\&\, B^0) \supset \neg A$ |

| | | |
|---|---|---|
| $A^0 \supset (A \supset \neg\neg A)$ | $A \vee (A \supset B)$ | $A \supset (\neg A \vee \neg\neg A)$ |
| $A \,\&\neg A \& A^0 \supset B$ | $A \supset (\neg\neg A \supset A)$ | $A^0, A \supset B \; \neg A \vee B$ |
| $A^0 \vdash (\neg A \supset A) \supset A$ | $A^+ \supset (A \equiv \neg\neg A)$ | $A^0 \vdash (A \supset \neg A) \supset \neg A$ |
| ${}^0A \vdash^0 (A \,\&\neg A)$ | $A^0 \vdash (A \;\&\; \neg A)^0$ | $(A \vee \neg A) \vee (A \vee \neg\neg A)$ |

THEOREM 1.6. *$V_0$ is finitely trivializable.*

PROOF. In $V_0$ there is a formula (of type $A \;\&\; \neg A \;\&\; A^0$) from which it is possible to obtain every formula.

## 2. The system $V_1$

This is the system corresponding to the case in which for every formula the law of excluded middle or the law of contradiction is valid but not both.

DEFINITION. $\neg^* A =_{df} A^0 \;\&\; (A \supset \neg A)$

The postulates of $V_1$ are 1-14 of $V_0$ plus:

15. $\neg^*\neg^* A \supset A$
16. $A \vee \neg^* A$

THEOREM 2.1. *$V_0$ is a proper subsystem of $V_1$.*

PROOF. It is obvious that $V_0$ is a subsystem of $V_1$, and it is easy to exhibit formulas (e.g. ${}^0A \vee A^0$) that are theorems of $V_1$ but not of $V_0$.

With obvious modifications, Theorems 1.1-1.4 and 1.6 are valid in $V_1$.

THEOREM 2.2. *Besides the schemas and rules of Theorem 1.5, we prove in $V_1$ among others the following schemas: ${}^0A \vee (A \;\&\; \neg A)$, $A \supset (\neg\neg A \supset A)$, $A \;\&\; \neg A \;\&\; A^0 \supset B$, $A \vee A^0$, ${}^0A \vee A^0$.*

## 3. The system $V_2$

This is the system corresponding to the case in which the law of contradiction is valid but not the law of excluded middle.

DEFINITION. $\neg^* A =_{df} A \supset \neg A$.

The postulates of $V_2$ are 1-9 of $V_1$ plus:

10. ${}^0A \,\&(A \supset B)\&(A \supset \neg B) \supset \neg A$
11. ${}^0A \;\&\; {}^0B \supset^0 (A \supset B)\&^0(A \;\&\; B)\&^0(A \vee B)$
12. ${}^0A \supset^0 (\neg A)$
13. $\neg(A \;\&\; \neg A)$
14. $\neg^*\neg^* A \supset A$.

Theorem 3.1. *$V_1$ is a proper subsystem of $V_2$ and of $C_0$.*

Theorems 1.1-1.4 and 1.6, with obvious modifications, are valid in $V_2$.

Theorem 3.2. *Besides the schemas and rules of Theorem 1.5 and 2.2, we prove in $V_2$, for example, the following:*

| | | |
|---|---|---|
| $A \supset \neg\neg A$ | $A \;\&\; \neg A \supset B$ | $\neg(A \supset \neg A) \supset A \;\&\; \neg\neg A$ |
| $\neg({}^0A \supset \neg A) \supset A$ | $A \equiv^0 A \;\&\; \neg\neg A$ | $\neg^*\neg^* A \supset \neg\neg A$ |
| $\neg A \supset \neg^* A$ | $\neg^*\neg^* A \supset A$ | *If* $\vdash A$, *then* $\vdash^0 A$ |
| ${}^0(A \;\&\; \neg A)$ | $\neg(A \supset B) \supset A$ | |

## 4. Duality between $V_2$ and $C_1$

Let us consider the system $C_1$ axiomatized as follows: the postulates of *LPC* (classical positive logic) plus:

| | |
|---|---|
| I) $B^0 \,\&(A \supset B)\&(A \supset \neg B) \supset \neg A$ | III) $\neg\neg A \supset A$ |
| II) $A^0 \;\&\; B^0 \,\&(A \supset B)^0\&(A \;\&\; B)^0\&(A \vee B)^0$ | IV) $A \vee \neg A$ |

(of course, Peirce's Law is not independent of the other axioms).

To establish a certain kind of duality between $V_2$ and $C_1$, we change the axiomatics of $V_2$ to the equivalent one: the postulates of *LPC*, plus:

| | |
|---|---|
| I′) $\neg A \supset (A \supset B)\&(A \supset \neg B)\& \, B^0$ | III′) $A \supset \neg\neg A$ |
| II′) ${}^0A \;\&\; {}^0B \supset^0 (A \supset B) \;\&\; {}^0(A \;\&\; B)\& \; {}^0(A \vee B)$ | IV′) $\neg(A \;\&\; \neg A)$ |

Now we say that I and I′, as well as III and III′ are "duals" because one is obtained from the other by reversing the order of the respective implication; and we say that II and II′, as well as IV and IV′, are "duals" because one is obtained from the other by interchanging $\neg(A \;\&\; \neg A)$ and

$A \vee \neg A$. But, in spite of this fact, it seems that it is impossible to obtain a general theorem of duality between $V_2$ and $C_1$. As an example of this conjecture one may observe the following:

$\vdash_{C_1} \neg(A \,\&\, \neg B) \supset \neg A \vee B \quad \nvdash_{V_2} \neg A \vee B \supset \neg(A \,\&\, \neg B)$
$\nvdash_{C_1} A \supset \neg(A \supset \neg A) \quad \vdash_{V_2} \neg(A \supset \neg A) \supset A.$

Nonetheless, some other examples of "duality" between $V_2$ and $C_1$ may be seen in the following table:

| $V_2$ | $C_1$ |
|---|---|
| $\neg^* A \supset \neg A$ | $\neg A \supset \neg^* A$ |
| $\neg\neg A \supset \neg^*\neg^* A$ | $\neg^*\neg^* A \supset \neg\neg A$ |
| $A^0 \supset (A \supset \neg\neg A)$ | ${}^0A \supset (\neg\neg A \supset A)$ |
| $\neg^*\neg A \supset A$ | $\neg\neg^* A \supset A$ |
| $A^0 \supset (\neg A)^0$ | ${}^0A \supset^0 (\neg A)$ |
| $(A \,\&\, \neg A)^0$ | ${}^0(A \,\&\, \neg A)$ |
| $(A^0)^0$ | ${}^0(A^0)$ |

## References

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