Bulletin of the **S**ection of **L**ogic
Volume 8/3 (1979), pp. 139–144

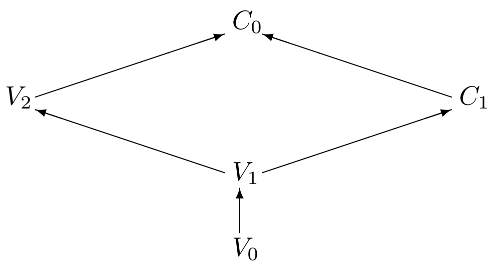
reedition 2010 [original edition, pp. 139–144]

Ayda I. Arruda and Elias H. Alves

A SEMANTICAL STUDY OF SOME SYSTEMS OF VAGUENESS LOGIC

1. Introduction

In [1] we have characterized four types vagueness related to negation, and constructed the corresponding propositional calculi adequate to formalize each type of vagueness. The calculi obtained were named V_0, V_1, V_2 and C_1 (C_1 is the first system of the hierarchy of paraconsistent logic of da Costa [2]). The relations among these calculi and the classical propositional calculus C_0 can be represented in the following diagram, where the arrows indicate that a system is a proper subsystem of the other



In this paper we present a two-valued semantics for each of these systems. The semantics used here is the Henkin-style semantics which has proven fruitful in treating other paraconsistent logics (see, fort example, [3]).

The terminology and notations here are those of [1]). The postulates of V_0 are the following (with the definitions: ${}^0A =_{df} A \vee \neg A$, $A^0 =_{df} \neg(A \,\&\, \neg A)$, $\neg^* A =_{df} A \supset (\neg A \,\&\, A^0)$):

1. $A \supset (B \supset A)$
2. $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
3. $A, A \supset B/B$
4. $A \supset (B \supset A \& B)$
5. $A \& B \supset A$
6. $A \& B \supset B$
7. $A \supset A \vee B$
8. $B \supset A \vee B$
9. $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
10. ${}^0A \& B^0 \& (A \supset B) \& (A \supset \neg B) \supset \neg A$
11. $A^0 \& B^0 \supset (A \supset B)^0 \& (A \& B)^0 \& (A \vee B)^0$
12. ${}^0A \& {}^0B \supset^0 (A \supset B) \& {}^0(A \& B) \& {}^0(A \vee B)$
13. $A^0 \supset (\neg A)^0$
14. ${}^0A \supset^0 (\neg A)$
15. $\neg^*\neg^*A \supset A$

The postulates of V_1 are 1-14 of V_0 plus:

15. $\neg^*\neg^*A \supset A$
16. $A \vee \neg^*A$

where $\neg^*A$ is defined by $A^0 \& (A \supset \neg A)$.

The postulates of V_2 are 1-9 of V_1 plus:

10. ${}^0A \& (A \supset B) \& (A \supset \neg B) \supset \neg A$
11. ${}^0A \& {}^0B \supset^0 (A \supset B) \& {}^0(A \& B) \& {}^0(A \vee B)$
12. ${}^0A \supset^0 (\neg A)$
13. $\neg(A \& \neg A)$
14. $\neg^*\neg^*A \supset A$

where $\neg^*A$ is defined by $A \supset \neg A$.

2. The semantics of V_0

Let $\mathbb{F}$ denote the set of formulas of V_0. Γ will designate any subset of $\mathbb{F}$. The set $\{A \in \mathbb{F} : \Gamma \vdash A\}$ will be denoted by $\overline{\Gamma}$.

Definition. The set Γ of formulas is said to be trivial in V_0 if $\mathbb{F} = \overline{\Gamma}$; otherwise, Γ is called nontrivial.

DEFINITION. Γ is a maximal nontrivial set if it is nontrivial and, for all A, if $A \notin \Gamma$, then $\Gamma \cup \{A\}$ is trivial.

THEOREM 1. *If Γ is maximal nontrivial, then:*

$\Gamma \vdash A \Leftrightarrow A \in \Gamma$	$A \in \Gamma$ *or* $\neg^* A \in \Gamma$	$\neg A \,\&\, A^0 \in \Gamma \Rightarrow \neg A \notin \Gamma$
$A \in \Gamma \Rightarrow \neg^* A \notin \Gamma$	$\vdash A \Rightarrow A \in \Gamma$	$\neg A \,\&\, A^0 \in \Gamma \Rightarrow A \notin \Gamma$
$\neg^* A \in \Gamma \Rightarrow A \notin \Gamma$		$A, A \supset B \in \Gamma \Rightarrow B \in \Gamma.$

DEFINITION. A valuation for V_0 is a function $\vartheta : \mathbb{F} \to \{0, 1\}$ such that:

1. $\vartheta(A \supset B) = 1 \Leftrightarrow \vartheta(A) = 0$ or $\vartheta(B) = 1$
2. $\vartheta(A \,\&\, B) = 1 \Leftrightarrow \vartheta(A) = \vartheta(B) = 1$
3. $\vartheta(A \vee B) = 1 \Leftrightarrow \vartheta(A) = 1$ or $\vartheta(B) = 1$;
4. $\vartheta(A^0) = \vartheta(B^0) = 1 \Rightarrow \vartheta((A \supset B)^0) = \vartheta((A \,\&\, B)^0) = \vartheta((A \vee B)^0) = 1$;
5. $\vartheta({}^0A) = \vartheta({}^0B) = 1 \Rightarrow \vartheta({}^0(A \supset B)) = \vartheta({}^0(A \,\&\, B)) = \vartheta({}^0(A \vee B)) = 1$;
6. $\vartheta(A^0) = 1 \Rightarrow \vartheta((\neg A)^0) = 1$
7. $\vartheta({}^0A) = 1 \Rightarrow \vartheta({}^0(\neg A)) = 1$;
8. $\vartheta(A) = \vartheta(\neg A) = 1 \Rightarrow \vartheta(A^0) = 0$.

LEMMA. *If ϑ is a valuation for V_0 then $\vartheta(A) = 1 \Leftrightarrow \vartheta(\neg^* A) = 0$.*

DEFINITION. The formula A is valid in V_0 if for each valuation ϑ, $\vartheta(A) = 1$. A valuation ϑ is a model of the set Γ if $\vartheta(A) = 1$ for all formulas $A \in \Gamma$. If each model of Γ is a model of $\{A\}$ we write $\Gamma \models A$. (In particular, $\models A$ means that A is valid.)

THEOREM 2 (THEOREM OF SOUNDNESS). $\Gamma \vdash A \Rightarrow \Gamma \models A$.

PROOF. By induction on the length of a deduction of A from Γ.

LEMMA. *If Γ is nontrivial, then Γ is contained in a maximal nontrivial set.*

PROOF. By an obvious adaptation of the corresponding classical theorem.

LEMMA. *Every maximal nontrivial set of formulas has a model.*

PROOF. We define the function $\vartheta : \mathbb{F} \to \{0, 1\}$ as follows: for every formula A, if $A \in \Gamma$, then $\vartheta(A) = 1$, otherwise $\vartheta(A) = 0$. Then we prove that ϑ is a valuation for V_0.

THEOREM 3 (THEOREM OF COMPLETENESS). $\Gamma \models A \Rightarrow \Gamma \vdash A$.

PROOF. Consequence of the preceding lemmas.

As an application of the semantics we can prove the following theorem.

THEOREM 4. *In V_0, the following schemas (among others) are not valid:*

$(A \vee \neg A) \vee \neg(A \ \& \ \neg A)$	$A \vee \neg A$	$\neg(A \ \& \neg A)$	${}^0A \supset (\neg\neg A \supset A)$
${}^0A \vee (A \ \& \ \neg A)$	$A \vee A^0$	$\neg A \vee A^0$	${}^0A \ \& \ A^0$.

3. The semantics of the system V_1

A semantics for the system V_1 can be obtained by strengthening clause 8 of Definition 3, in the following way:

8′. $\vartheta(A) = \vartheta(\neg A) = 1 \Leftrightarrow \vartheta(A^0) = 0$.

With this definition of valuation it is easy to prove the theorems of soundness and completeness. We need only to show that function ϑ which appears in the proof of the last lemma satisfies the clause 8′ of the definition of valuation for V_1, and this is not difficult.

As an application of the semantics we can prove the following result:

THEOREM 5. *In V_1, the following schemas (among others) are not valid: $A \supset \neg\neg A$, $\neg\neg A \supset A$, $A \vee \neg A$, $\neg(A \ \& \ \neg A)$.*

THEOREM 6. *V_2 is a proper subsystem of V_1.*

PROOF. It is enough to verify that axiom 15 of V_0 is a theorem of V_1, and that there are formulas which are not valid in V_0 are theorems of V_1 (for instance, ${}^0A \vee A^0$).

4. The semantics of the system V_2

V_2 is obtained from V_1 by adding as new axiom the law of contradiction, $\neg(A \ \& \ \neg A)$. A semantics for the system V_2 can be obtained by adding in the definition of valuation for V_1, the following clause:

9. $\vartheta(A) = 1 \Rightarrow \vartheta(\neg A) = 0$.

With this definition of valuation, it is easy to prove the theorems of soundness and completeness. We will only show that the new axiom is valid in V_2. Suppose not: then there is a valuation ϑ, such that $\vartheta(\neg(A \ \& \ \neg A)) =$

0. Then, by clause $8'$, we have $\vartheta(A) = \vartheta(\neg A) = 1$, which contradicts clause 9.

As an application of the semantics, we have the following result:

THEOREM 7. *In V_2, the following schemas (among others) are not valid:* $\neg\neg A \supset A$, $A \vee \neg A$, $\neg A \vee \neg B \supset \neg(A \;\&\; B)$, $(A \supset B) \supset \neg A \vee B$.

THEOREM 8. *V_1 is a proper subsystem of V_2.*

PROOF. It is immediate, taking into account the construction of V_2 and the fact that $\neg(A \;\&\; \neg A)$ is not a theorem of V_1 (see Theorem 5).

5. Final remarks

Observe that the semantics of V_1 can also be extended in order to obtain a semantics for the system C_1 of da Costa. This can be done by adding the following clause in the definition of valuation:

$9'$. $\vartheta(A) = 0 \Rightarrow \vartheta(\neg A) = 1$.

It is easy to show that this definition of valuation is equivalent to that formulated, for example, in [3].

Finally, we observe that a decision method for each of the system V_0, V_1, and V_2 can be obtained by an adaptation of the method proposed in da Costa and Alves [3].

References

[1] A. I. Arruda and E. H. Alves, *Some remarks on the logic of vagueness*, **Bul. of the Section of Logic**, Polish Academy of Sciences..

[2] N. C. A. da Costa, *On the theory of inconsistent formal systems*, **Notre Dame J. of Formal Logic** XI (1974), pp. 497–510.

[3] N. C. da Costa and E. H. Alves, *A semantical analysis of the calculi C_n*, **Notre Dame Journal of Formal Logic** XVIII (1977), pp. 621–630.

Departamento de Matemática
Universidade Estadual de Campinas
13100 Campinas, Brazil

and

Centro de Lógica, Epistemologia e História da Ciência
Universidade Estadual de Campinas
12100 Campinas, Brazil