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NOT ALL REPRESENTABLE CYLINDRIC ALGEBRAS ARE NEAT REDUCTS

This is a solution to the problem raised in the monograph [1] on algebraic logic.

Cylindric algebras are the algebraic counterparts of First Order Logic as was explained in the monograph [1] of Henkin, Monk, and Tarski, and also in [2], [3], and [4]. A cylindric algebra is representable if it corresponds to some logical system in a strong sense, cf. Theorem 4.2 and Definition 6.2 in [2] and 1.1.13 of [1]. (see also the remark preceding Corollary 2 in the present note). It was shown in [1], cf. Corollary 3.14 and Corollary 3.18 of [2], that the class R_α od all representable cylindric algebras of dimension α coincides with the class $S\ Nr_\alpha\ CA_{\alpha+\omega}$ of all subalgebras of neat reducts. Here $Nr_\alpha CA_{\alpha+\omega}$ denotes the class of all neat reducts, see 2.6.28 of [1]. Therefore neat reducts are strongly related to algebraic versions of logical systems, cf. 2.6.26 of [1]. (See the remarks preceding Corollary 2 in the present note). The question arose *how close* this relation is: Problem 2.11 on p. 464 of [1] is the question *whether* the class $Nr_\alpha CA_\beta$ of all α-dimensional neat reducts of β-dimensional cylindric algebras is closed under the formation of subalgebras and homomorphic images or not.

The Theorem below formulates an answer to this question. We shall use the notations of [1], e.g. if K is a class of algebras then $S\ K$ and $H\ K$ are the classes of all subalgebras of elements of K and all homomorphic images of elements of K, respectively.

Theorem. *For arbitrary ordinals $\alpha < \beta$, (i) and (ii) below hold.*

(i) $H\ Nr_\alpha CA_\beta = Nr_\alpha CA_\beta$

(ii) $S\ Nr_\alpha CA_\beta \neq Nr_\alpha CA_\beta$ *if and only if* $1 < \alpha$.

Corollary 1.

(i) $Up\ Nr_{\alpha}CA_{\beta} = Nr_{\alpha}CA_{\beta}$ *for every* $\alpha \leqslant \beta$.

(ii) $CA_1 = Nr_1 CA_{\beta}$ *for every ordinal* $\beta \geqslant 1$.

(iii) $SP\ Lf_{\omega}\ Nr_{\omega}CA_{\omega+\omega}$.

Part (iii) above implies that there exist theories of Typeless Logic which are not neat reducts in their algebraic form, cf. Theorem 4.2 of [2].

Let Lv denote the class of all locally i-finite cylindric set algebras as defined in [2] p. 18–19, in [4] p. 10, and [3]. It was proved in [2] and in more detail in [3] that Lv_{ω} is *the* class of all cylindric algebras corresponding to models of classical first logic. The elements of $SP\ Lv_{\omega}$ therefore correspond to classes of (not necessarily similar) models, cf. [2] and [3].

Corollary 2. $SP\ Lv_{\omega} \not\subseteq Nr_{\omega}CA_{\omega+\omega}$.

I.e. there exists a cylindric algebra $\underline{A}$ corresponding to a class M of models of first order logic such that $\underline{A} \notin Nr_{\omega}CA_{\omega+\omega}$, i.e. $\underline{A}$ is not a neat reduct (despite of its logical origin).

A detailed investigation of Lv_{ω} can be found in [5] p. 14–19 and in [4]. The relationship between Lv_{ω} and Logic was also explained there.

In view of these references, Corollary 2 above implies that in Typeless Logic there are concepts definable by using auxiliary variable symbols which are not definable without auxiliary variables.

References

[1] L. Henkin, J. D. Monk, A. Tarski, **Cylindric Algebras, Part I**, North-Holland 1971.

[2] H. Andréka, T. Gergely, I. Németi, *On Universal Algebraic Construction of Logics*, **Studia Logica** XXXVI 1-2, (1977), pp. 10–47.

[3] I. Németi, I. Sain, *Connections between Algebraic Logic and Initial Algebra Semantics of CF languages*, submitted to Proc. Coll. Logic in Progr., Salgótarján 1978.

[4] H. Andréka, I. Németi, *A simple, purely algebraic proof of the completeness of some first order logics*, **Algebra Universalis**, Vol.5, (1975), pp. 8- 15.

[5] H. Andréka, T. Gergely, I. Németi, **Purely algebraical construction of first order logics**, Preprint KFKI-73-71, (1973).

[6] L. Henkin, J. D. Monk, A. Tarski, **Cylindric Set Algebras and Related Structures**, Manuscript to be published.

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