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ON $\{\leftrightarrow, \neg\}$-REDUCT OF THE INTUITIONISTIC CONSEQUENCE OPERATION

Dedicated to
Professor Roman Suszko

This is an abstract of a paper which will be submitted to Studia Logica.

In a letter to the third author Professor Roman Suszko asked for a basis of the consequence operation of equivalence and weak negation i.e. the smallest consequence operation in the language FM whose only connectives are $\leftrightarrow$ (equivalence) and $\neg$ (negation) satisfying the following conditions for every $X \subseteq FM$, $\alpha, \beta \in FM$:

($\leftrightarrow$) $C(X \cup \{\alpha\}) = C(X \cup \{\beta\})$ iff $\alpha \leftrightarrow \beta \in C(X)$,
($\neg$) $C(X \cup \{\alpha\}) = FM$ iff $\neg\alpha \in C(X)$.

In this paper we shall prove that the following rules (a1),. . .,(a8) constitute a basis of the consequence operation in question:

(a1) $\vdash \alpha \leftrightarrow \alpha$,
(a2) $\vdash (((\alpha \leftrightarrow \beta) \leftrightarrow \gamma) \leftrightarrow \gamma) \leftrightarrow ((\alpha \leftrightarrow \gamma) \leftrightarrow (\beta \leftrightarrow \gamma))$,
(a3) $\vdash (((\alpha \leftrightarrow \beta) \leftrightarrow ((\alpha \leftrightarrow \gamma)) \leftrightarrow ((\alpha \leftrightarrow \gamma) \leftrightarrow \gamma)) \leftrightarrow (\alpha \leftrightarrow \beta)$,
(a4) $\alpha, \alpha \leftrightarrow \beta \vdash \beta$,
(a5) $\alpha \vdash (\alpha \leftrightarrow \beta) \leftrightarrow \beta$,
(a6) $\vdash (((\neg\alpha) \leftrightarrow \beta) \leftrightarrow \beta) \leftrightarrow (\neg\alpha)$,
(a7) $\vdash ((\neg\alpha) \leftrightarrow \alpha) \leftrightarrow ((\neg\beta) \leftrightarrow \beta)$,
(a8) $(\neg\alpha) \leftrightarrow \alpha \vdash \beta$.

It is known that the smallest consequence operation in FM satisfying $(\leftrightarrow)$ and $(\neg)$ is just the intuitionistic consequence operation C_{INT} restricted to the language FM (see [2]). Thus our answer to the question of Professor Roman Suszko results in a basis of $\{\leftrightarrow, \neg\}$-reduct of the intuitionistic consequence operation i.e. a basis of the consequence operation C in FM such that $C(X) = FM \cap C_{INT}(X)$ for every $X \subseteq FM$. Let us also note that a basis of $\{\leftrightarrow\}$-reduct of the intuitionistic consequence operation can be obtained by taking the rules (a1),…,(a5) (comp. [1], [3]).

In order to make the notation more readable we adopt the convention of ignoring the equivalence sign $\leftrightarrow$ in formulas of FM. Moreover we assume that $\neg$ binds stronger then $\leftrightarrow$ and formulas with lacking parentheses are to be associated to the left. Thus for example the rules (a1),…,(a8) can be written as follows:

(a1) $\vdash \alpha\alpha$,
(a2) $\vdash \alpha\beta\gamma\gamma(\alpha\gamma(\beta\gamma))$,
(a3) $\vdash \alpha\beta(\alpha\gamma\gamma)(\alpha\gamma\gamma)(\alpha\beta)$,
(a4) $\alpha, \alpha\beta \vdash \beta$,
(a5) $\alpha \vdash \alpha\beta\beta$,
(a6) $\vdash \neg\alpha\beta\beta\neg\alpha$,
(a7) $\vdash \neg\alpha\alpha(\neg\beta\beta)$,
(a8) $\neg\alpha\alpha \vdash \beta$.

Let C be the consequence operation in FM determined by the basis (a1),…,(a8) and let C^- be the weakening of C determined by (a1),…(a7). The following rules can be derived from (a1),…,(a7) and so they are rules of the consequence operation C^-:

(r1) $\alpha\beta \vdash \beta\alpha$,
(r2) $\alpha\beta, \beta\gamma \vdash \alpha\gamma$,
(r3) $\vdash \alpha\beta(\beta(\alpha\alpha\alpha))$,
(r4) $\vdash \alpha\beta(\beta\alpha)$,
(r5) $\vdash \alpha\alpha(\beta\beta)$,
(r6) $\vdash \neg\alpha\alpha\neg(\alpha\alpha)$,
(r7) $\alpha\beta \vdash \alpha\gamma(\beta\gamma)$,
(r8) $\alpha\beta \vdash \gamma\alpha(\gamma\beta)$,
(r9) $\alpha\beta \vdash \neg\alpha\neg\beta$,
(r10) $\vdash \neg\alpha\alpha\beta\gamma\gamma\neg\beta$.

It can be proved by using techniques of [1] that the consequence operation C^- satisfies the condition ($\leftrightarrow$) i.e. we have the following:

LEMMA 1. *For every* $X \subseteq FM$, $\alpha, \beta \in FM : C^-(X \cup \{\alpha\}) = C^-(X \cup \{\beta\})$ *iff* $\alpha\beta \in C^-(X)$.

Let ε be an arbitrary fixed formula of FM, we introduce the following abbreviations:

$$\mathbf{1} = \varepsilon\varepsilon, \quad \mathbf{0} = \neg\varepsilon\varepsilon.$$

Then we have the following:

LEMMA 2. *For every* $X \subseteq FM$ *and* $\alpha \in FM$ *the following conditions hold:*

(i) If $\mathbf{0} \in C(X)$, *then* $\mathbf{0} \in C^-(X)$,
(ii) If $\mathbf{0} \in C(X)$, *then* $C(X) = C^-(X)$,
(iii) If $\mathbf{0} \in C(X \cup \{\alpha\})$, *then* $\{\mathbf{0}\alpha, \mathbf{0}\} \cap C^-(X) \neq \emptyset$.

Now we are in the position to prove the following:

LEMMA 3. *For every* $X \subseteq FM$, $\alpha, \beta \in FM$ *the following conditions hold:*

(i) $C(X \cup \{\alpha\}) = C(X \cup \{\beta\})$ *iff* $\alpha\beta \in C(X)$,
(ii) $C(X \cup \{\alpha\}) = FM$ *iff* $\neg\alpha \in C(X)$.

PROOF. It is easy to see that the only difficulty is in proving the implication from the left to the right in the condition (i). Let us suppose that $C(X \cup \{\alpha\}) = C(X \cup \{\beta\})$. First consider the case when $\mathbf{0} \in C(X \cup \{\alpha\})$. Then by virtue of Lemma 2 (i): $\mathbf{0} \in C^-(X \cup \{\alpha\})$ and $\mathbf{0} \in C^-(X \cup \{\beta\})$. Next by Lemma 2 (iii) it follows that $\{\mathbf{0}\alpha, \mathbf{0}\} \cap C^-(X) \neq \emptyset$, $\{\mathbf{0}\beta, \mathbf{0}\} \cap C^-(X) \neq \emptyset$. Consequently $\mathbf{0} \in C^-(X)$ or $\{\mathbf{0}\alpha, \mathbf{0}\beta\} \subseteq C^-(X)$. In either case $\alpha\beta \in C(X)$. Now let us consider the case when $\mathbf{0} \notin C(X \cup \{\alpha\})$. Then Lemma 2 (ii) yields that $C(X \cup \{\alpha\}) = C^-(X \cup \{\alpha\})$, $C(X \cup \{\beta\}) = C^-(X \cup \{\beta\})$ and by Lemma 1 we get that $\alpha\beta \in C^-(X) \subseteq C(X)$.

THEOREM. *The consequence operation* C *determined by the rules (a1),…, (a8) is the smallest consequence operation in* FM *satisfying the condition* ($\leftrightarrow$) *and* ($\neg$).

PROOF. In view of Lemma 3 it suffices to show that (a1),…,(a8) are the rules of every consequence operation satisfying ($\leftrightarrow$) and ($\neg$) which is very easy.

COROLLARY. *$\{\leftrightarrow, \neg\}$-fragment of the intuitionistic propositional logic can be axiomatized by (a1),…,(a7).*

REMARK. A set of rules equivalent to (a1),…,(a7) can be obtained by adopting (r10) in place of (a6) and (a7). A basis for $\{\leftrightarrow, \neg\}$-reduct of the classical consequence operation can be obtained by taking (a1),…,(a8) and

(a9) $\vdash \neg\neg\alpha\alpha$.

Analogously $\{\leftrightarrow, \neg\}$-reduct of the classical propositional calculus can be axiomatized by (a1),…,(a7), (a9).

References

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