Bulletin of the **S**ection of **L**ogic
Volume 8/4 (1979), pp. 170–174
reedition 2010 [original edition, pp. 170–176]

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REFERENTIAL MATRIX SEMANTICS FOR PROPOSITIONAL CALCULI

An invited lecture delivered at 6th International Congress of Logic Methodology and Philosophy of Science, Hannover August 22-August 29, 1979. An extended version will be published in the Proceedings the Congress.

1. Preliminaries

Let $\underline{L}$ be a propositional language, and let $F_1, \ldots, F_n$ be all propositional connectives it involves. We shall denote the set of all formulas of $\underline{L}$ as L, and identify $\underline{L}$ with the algebra of formulas $(L, F_1, \ldots, F_n)$ freely generated by the propositional letters $p_1, \ldots, p_i, \ldots$.

Assume that T is a non-empty set of indices (points of reference, possible worlds) and assume that a certain set H of functions of the form $v : L \times T \to \{0, 1\}$ is selected to play the role of admissible valuations for $\underline{L}$ (i.e. the valuations that conform to the intended meaning of F_i's). $v(\alpha, t) = 1$ $(= 0)$ reads: the formula α is true (is false) at the point t under v.

The set H determines uniquely the consequence operation Cn_H on $\underline{L}$ defined by

(H) $\alpha \in Cn_H(X)$ iff for each $v \in H$ and each $t \in T$, $v(\alpha, t) = 1$ whenever $v(\beta, t) = 1$ for all $\beta \in X$.

If $H = Hom(\underline{l}, \underline{A})$, for some algebra $\underline{A}$ similar to $\underline{L}$, then I shall write $Cn_{\underline{A}}$ instead of Cn_H, and occasionally denote T as $T_{\underline{A}}$. Observe that if $H = Hom(\underline{L}, \underline{A})$, the elements of $\underline{A}$ are functions of the form $r : T \to \{0, 1\}$. Let me call such algebras *referential*.

Each couple $(\underline{L}, Cn_{\underline{A}})$, where $\underline{A}$ is a referential algebra, will be referred to as a *referential* (*referentially truth-functional*) *propositional logic*. In general, by a *propositional* logic I shall understand a couple $(\underline{L}, C)$ where C is a structural consequence operation on $\underline{L}$, i.e. a function from $P(L)$ into $P(L)$ such that for all $X, Y \subseteq L$:

(C_1) $X \subseteq C(C(X)) \subseteq C(X) \subseteq C(X \cup Y)$,
(C_2) $eC(X) \subseteq C(eX)$, for all endomorphisms e of $\underline{L}$.

As known, the class of all structural consequence operations on $\underline{L}$ forms a complete lattice with the lattice ordering $\leqslant$ defined by

$$C_1 \leqslant C_2 \text{ iff } C_1(X) \subseteq C_2(X), \text{ for all } X \subseteq L.$$

Let $\mathbb{K}$ be a class of referential algebras similar to $\underline{L}$. We define $Cn_{\mathbb{K}} = inf\{Cn_A : A \in \mathbb{K}\}$. The logic $(\underline{L}, Cn_{\mathbb{K}})$ (the consequence $Cn_{\mathbb{K}}$) will be referred to as determined by $\mathbb{K}$.

THEOREM 1. *For each class $\mathbb{K}$ of referential algebras there exists a singular referential algebra A such that $Cn_{\mathbb{K}} = Cn_A$.*

COROLLARY. *A propositional logic is referential iff it is determined by a class of referential matrices.*

2. A syntactical criterion of referentiality

Call a propositional logic $(\underline{L}, C)$ *self-extensional* iff $C(\alpha) = C(\beta)$ implies that $C(\varphi(\alpha/p)) = C(\varphi(\beta/p))$ for all $\alpha, \beta, \varphi \in L$, and all propositional letters p. ($\varphi(\alpha/p)$ denotes the formula that results from φ by replacing every occurrence of p by α).

THEOREM 2. *A propositional logic is referential iff it is self-extensional.*

With positive Hilbert's Logic, Minmal Johansson's Logic, Intuitionistic Logic, and – of course – Classical Two-Valued Logic are easily seen to be self-extensional and thus referential. Logic with Constructible Falsity and Łukasiewicz's Many-Valued Logics may serve as examples of nonreferential Logics. Referentiality of Modal Logics depends on how the consequence operation of the logic in question is defined.

3. Modal Logics

Let $\underline{L_\Box} = (L_\Box, \to, \neg, \Box)$ be a proposition language of the similarity type $(2,1,1)$. A logic $(\underline{L_\Box}, C)$ will be said to be *modal* iff

i. $C(\emptyset)$ contains all classical tautologies expressed in terms of $\to$ and $\neg$.

ii. For each X, $C(X)$ is closed under Modus Ponens.

By a *modal consequence* I shall mean the consequence of a modal logic, and by a *modal system* a set of formulas of $\underline{L_\Box}$ of the form $C(\emptyset)$, where C is a modal consequence.

Observe that for each modal system M there exists a least modal consequence C such that $M = C(\emptyset)$. The consequence C will be denoted as C_M, and the modal logic $(\underline{L_\Box}, C_M)$ will be called *associated* with M.

Let us restrict our further discussion to the following well known modal systems: $E, C, K, T, B, S4, S5$ usually referred to is modal logics, cf. e.g. K. Segerberg [2]/

Call a referential algebra $\underline{A}$ similar to $\underline{L_\Box}$ *standard* iff the operations $\to_{\underline{A}}$ of $\underline{A}$ satisfy the following two conditions:

$(\to)$ $(r_1 \to_{\underline{A}} r_2)(t) = 1$ iff $r_1(t) = 0$ or $r_2(t) = 1$,

$(\neg)$ $(\neg r_1)(t) = 1$ iff $r_1(t) = 0$.

Given a standard algebra $\underline{A}$ define $P_{\underline{A}}(T_{\underline{A}})$ to be the set of all subsets of $T_{\underline{A}}$ of the form $\{t : r(t) = 1\}$, $r \in \underline{A}$. Furthermore define the *neighborhood function* $N_{\underline{A}} : T_{\underline{A}} \to P(P_{\underline{A}}(T_{\underline{A}}))$ of $\underline{A}$ and the *alternation relation* R_A of A as follows:

(N) $\{t : r(t) = 1\} \in N_{\underline{A}}(t_0)$ iff $\Box_{\underline{A}} r(t_0) = 1$

(R) $t_1 R_{\underline{A}} t_2$ iff $N_{\underline{A}}(t_1) \neq \emptyset$ and $t_2 \in \bigcap N_{\underline{A}}(t_1)$.

We shall say that a standard algebra $\underline{A}$ is:

– *regular* iff for each t in $T_{\underline{A}}$ either $N_{\underline{A}}(t) = \emptyset$ or $N_{\underline{A}}(t)$ is a filter,

– *normal* iff for each t in $T_{\underline{A}}$, $N_{\underline{A}}(t)$ is a filter,

– *reflexive symmetric, transitive* iff $R_{\underline{A}}$ is reflexive symmetric, transitive.

The concepts defined are obvious counterparts of some well known concepts applied in neighborhood semantics and relational semantics for modal logic. It is worth realizing then how referential algebras are related to neighborhood and relational frames.

Call a referential algebra $\underline{A}$ and a frame F equivalent iff the set of all admissible valuations of formulas of $\underline{A}$ in F coincides with $Hom(\underline{L}_\Box, \underline{A})$. One may easily show that for each frame F there exists a referential algebra $\underline{A}$ such that F and $\underline{A}$ are equivalent. At the same time, in order for the equivalence to take place $\underline{A}$ must be standard and moreover *full*, i.e. it must involve as its elements all functions from T_A into $\{0, 1\}$. (Observe that standard referential algebras can be viewed as Boolean frames and, in particular, full referential algebras can be viewed as complete atomic Boolean frames. The connection between Boolean frames and neighborhood frames was examined by M. Gerson [1].)

In view of the observation we have made it is somewhat surprising that the word "full" does not appear in the following theorem.

THEOREM 3. *$C_E, C_C, C_K, C_T, C_B, C_{S4}, C_{S5}$ are respectively determined by the class of all*

(E) *standard referential algebras,*
(C) *regular referential algebras,*
(K) *normal referential algebras,*
(T) *normal and reflexive referential algebras,*
(B) *normal, reflexive, and symmetric referential algebras,*
($S4$) *normal, reflexive, and transitive referential algebras,*
($S5$) *normal, reflexive, symmetric, and transitive referential algebras.*

COROLLARY. *All logics of the form $(\underline{L}_\Box, C)$ where C is one of the consequences considered in Theorem 3 are referential. Furthermore, $(\underline{L}_\Box, C_E)$ is the weakest referential logic in the sense that C_E is the infimum of all modal and referential consequences on $\underline{L}$.*

AN OPEN PROBLEM. Provide a syntactic characteristic of the consequence operations determined by the classes of full referential algebras of the same sort as those defined by clauses (E) – ($S5$) of Theorem 3.

4. Referential matrices

Let $\underline{L}$ be a propositional language. Let $\underline{A}$ be an algebra similar to $\underline{L}$ and let D be a non-empty family of subsets of $\underline{A}$. The couple $(\underline{A}, D)$ is called a *generalized* (or *ramified*) *logical matrix* for $\underline{L}$. When D consists of a single subset of $\underline{A}$, the matrix $(\underline{A}, D)$ becomes a logical matrix for $\underline{L}$ in the standard Łukasiewicz-Tarski sense.

Define

$(M) \alpha \in Cn_{(\underline{A},D)}(X)$ iff for each $h \in Hom(\underline{L}, A)$ and for each
$I \in D$, $h\alpha \in I$ whenever $hX \subseteq I$.

The operation $Cn_{(\underline{A},D)}$ is easily seen to be a structural consequence on $\underline{L}$. The important thing about logical matrices is that for each structural consequence operation C on $\underline{L}$ there exist a generalized logical matrix $(\underline{A}, D)$ such that $C = Cn_{(\underline{A},D)}$ (cf. e.g. [3]). This explains why logical matrices should be considered as one of the most important technical tools for examinings propositional logics. At the same time it is worth-realizing that some important philosophical ideas can be expressed in terms of logical matrices.

There is an quite clear correspondence between neighborhood and relational frames on one hand and referential algebras on the other. Now, given an referential algebra $\underline{A}$ put $D_t = \{r \in \underline{A} : r(t) = 1\}$, $t \in T_A$ and define $D = \{D_t : t \in T_A\}$. One may easily verify that

$$(*) \qquad Cn_{\underline{A}} = Cn_{(\underline{A},D)}.$$

Thus to each referential algebras there correspond a logical matrix which I propose to call a *referential logical matrix* equivalent to it in the sense that $(*)$ holds true.

References

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