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ON STANDARD CONSEQUENCE OPERATIONS IN THE IMPLICATIONLESS LANGUAGE

This is an abstract of the paper submitted to Reports on Mathematical Logic.

This paper is a continuation of [1] and so all the concepts and notations introduced in [1] will be used freely. We shall consider here the lattice of all standard consequence operations in the implicationless language intermediate between the intuitionistic and the classical consequence operation. The problem of characterizing this lattice was suggested by Prof. Wolfgang Rautenberg. We shall show that the lattice in question is denumerable and it is isomorphic to a chain of type $(\omega + 1)^*$.

I wish to express my deep gratitude to Professor Andrzej Wroński for his help during the research.

For every $n = 0, 1, 2, \ldots, \omega$ let C^n be the structural consequence operation in $\mathcal{F}$ determined by the intermediate logic L_n (i.e. for every $X \subseteq F$, $F\alpha \in F$, $\alpha \in C^n(X)$ iff α can be derived from $L_n \cup X$ by means of the rule of detachment). For every $n = 0, 1, 2, \ldots, \omega$ let C_0^n be the consequence operation in $\mathcal{F}_0$ resulting by restricting of C^n to F_0 (i.e. for every $X \subseteq F_0$, $C_0^n(X) = C^n(X) \cap F_0$).

Lemma 1.

(i) *For every* $n = 0, 1, 2, \ldots, \omega$ *and for every* $\alpha, \beta \in F_0$, $\alpha \in C_0^n(\beta)$ *iff* $\beta \to \alpha \in L_n$.

(ii) *If* $\alpha \in F$ *is of the form* $\alpha_1 \to \alpha_2$ *where* $\alpha_1, \alpha_2 \in F_0$ *then for every* $n = 0, 1, 2, \ldots, \alpha \in L_n$ *iff* $\alpha \in E(\mathcal{R}^n \oplus)$ *and* $\alpha \in L_\omega$ *iff* $\alpha \in \bigcap_{n=0}^{\infty} E(\mathcal{R}^n \oplus)$.

For every $n = 0, 1, 2, \ldots, \omega$ by Θ_n we denote the congruence relation of the absolutely free algebra $\mathcal{F}$ defined as follows: for every $\alpha, \beta \in F$, $\alpha \equiv \beta(\Theta_n)$ iff $\alpha \to \beta$, $\beta \to \alpha \in L_n$. Then we have the following

LEMMA 2.

(i) *For every* $n = 0, 1, 2, \ldots$ *there exists* $\{\wedge, \neg\}$*-embedding of the algebra* $\mathcal{R}^n\oplus$ *into the quotient algebra* $\mathcal{F}/\Theta_n$.

(ii) *For every* $n = 0, 1, 2, \ldots$ *the algebra* $\mathcal{R}^n\oplus$ *is* $\{\wedge, \neg\}$*-embedable into the quotient algebra* $\mathcal{F}/\Theta_\omega$.

For every $n = 0, 1, 2, \ldots, \omega$ let $D^n = \{C | C$ is a standard consequence operation in $\mathcal{F}_0$ such that $C_0^\omega \leqslant C$ and $C(\emptyset) = C_0^n(\emptyset)\}$. The following lemma states that for every $n = 0, 1, 2, \ldots, \omega$ the set D^n contains exactly one element, namely the consequence operation C_0^n:

LEMMA 3.

(i) *For every* $n = 0, 1, 2, \ldots, \omega, C_0^n$ *is the greatest element of* D^n.

(ii) *For every* $n = 0, 1, 2, \ldots, \omega, C_0^n$ *is the smallest element of* D^n.

Now let D be the set of all standard consequence operations in $\mathcal{F}_0$ intermediate between the intuitionistic and the classical one (i.e. $D = \{C | C$ is a standard consequence operation in $\mathcal{F}_0$ such that $C_0^\omega \leqslant C \leqslant C_0^0\}$).

Using the lemmas above the reader will have no difficulty in proving the following result:

THEOREM.

(i) $D = \{C_0^n | n = 0, 1, 2, \ldots, \omega\}$

(ii) *For every* $n = 0, 1, 2, \ldots$ *the algebra* $\mathcal{R}^n\oplus$ *is strongly adequate for the consequence operation* C_0^n.

(iii) *The algebra* $\mathcal{D}\oplus$*, where* $\mathcal{D}$ *is the free* ω*-genereated Boolean algebra, is strongly adequate for the consequence operation* C_0^ω.

(iv) *For every* $n = 0, 1, 2, \ldots, \omega$*, the consequence operation* C_0^n *is structurally complete in the sense of [2].*

References

[1] E. Capińska, *On intermediate logics which can be axiomatized by means of implicationless formulas*, this **Bulletin**, pp. 197–201.

[2] W. A. Pogorzelski, *Structural completness of the propositional calculus*, **Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.** 19 (1971), pp. 349–351.

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