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A VARIETY BY A FINITE ALGEBRA WITH $2^{\aleph_0}$ SUBVARIETIES

Abstract

As it was indicated to me by Prof. A. Wroński the following problem was suggested by Prof. B. Jónsson: is every subvariety of a variety of a finite algebra generated by a finite algebra? In this paper we solve the problem in negative by construction of some finite algebra that generates variety having $2^{\aleph_0}$ subvarieties.

§1.

Throughout this paper $\mathcal{F} = (F, \circ)$ is assumed to be an absolutely free algebra of type $\langle 2 \rangle$ free generated by an infinite set of variables $x_0, x_1, \ldots$. Elements of $\mathcal{F}$, sometimes called terms, will be denoted by Greek letters $\alpha, \beta, \ldots$. To simplify the shape of terms we adopt the convention of associating to the right and ignoring the symbol of binary operation. For example, we shall write $(\alpha\beta)\gamma\delta\gamma$ instead of $(\alpha \circ \beta) \circ (\gamma \circ (\delta \circ \gamma))$. By $V(\alpha)$ we denote the set of all variables occurring in α, while $l(\alpha)$ is the variable occurring in α as first from the left.

Let O_1, O_2 be occurrences of some variables in the term α. We say that O_1 and O_2 are conjugate in α iff there exists a subterm of α of the form $\beta\gamma$ such that O_1 and O_2 are the first from the left occurrences of some variables in β and γ, respectively, or inversely. $C(\alpha)$ means the set of all unordered pairs of variables from the term α which have conjugate occurrences in α.

Let $a(n)$, $n < \omega$, be a sequence of natural numbers defined as follows: $a(O) = 1$ and $a(n+1) = 2 \cdot a(n) + 1$, for all $n \geq 0$. Let $\mathbb{K}_n$ $(n \geq 2)$ be the set of all terms α with the properties: (1) $V(\alpha) = \{x_1, x_2, \ldots, x_{a(n)}\}$, (ii) $l(\alpha) = x_1$ (iii) $C(\alpha)$, consists exactly of the following unordered pairs of variables: $\{x_i, x_{i+1}\}$ $(1 \le i \le a(n-1))$, $\{x_j, x_k\}$ $(a(n-1) < j \neq k \le a(n))$ and $\{x_1, x_{a(n-1)+1}\}$. Now, define $\mathbb{P}(n \ge 2)$ as the set of all terms from $\mathbb{K}_n$ in which there is some occurrence of x_2 being simultaneously conjugate with some occurrence of x_1 and of x_3.

For i, j such that $i < j$ define the term:

$$\gamma_i^j = \begin{cases} (x_i x_{i+1})(x_{i+2} x_i)(x_i x_{i+3}) \ldots (x_i x_j), & \text{whenever } j-1 \text{ is odd} \\ (x_i x_{i+1})(x_{i+2} x_i)(x_i x_{i+3}) \ldots (x_j x_i), & \text{whenever } j-1 \text{ is even} \end{cases}$$

For example, $\gamma_0^4 = (x_0x_1)(x_2x_0)(x_0x_3)(x_4x_0)$ and $\gamma_2^5 = (x_2x_3)(x_4x_2)(x_2x_5)$. Put $\delta_n = \gamma_{a(n-1)+1}^{a(n)}\ \gamma_{a(n-1)+2}^{a(n)} \cdots \gamma_{a(n)-1}^{a(n)}$, for all $n \ge 2$. Notice that $C(\delta_n)$ contains only pairs of the form $\{x_j, x_k\}$, where $a(n-1) < j \neq k \le a(n)$. Now, set $\alpha_n = x_1x_2x_3 \ldots x_{a(n-1}x_{a(n-1+1}x_1\delta_n$ and
$\beta_n = (x_1x_2)x_{a(n-1)+1}x_{a(n-1)} \ldots x_3x_2x_3 \ldots x_{a(n-1)}\delta_n$, for all $n \ge 2$.

LEMMA 1. *For any $n \ge 2$:*
(i) $\alpha_n, \beta_n \in \mathbb{K}_n$, (ii) $\alpha_n \in \mathbb{P}_n$ and $\beta_m \notin \mathbb{P}_n$.

The set of all identities which are valid in an algebra will be denoted by $Id(\mathcal{A})$, while Σ^*, Σ being a set of identities, will denote the class of all algebras similar to $\mathcal{F}$, in which all identities from Σ are valid.

§2.

Let $\mathcal{A} = (\{0,1,2,3,4,5,6\}, \circ)$ be an algebra of type $\langle 2 \rangle$, where the binary operation is given as follows:

$\circ$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0
2	0	2	2	2	2	2	2
3	0	0	5	3	3	3	3
4	0	0	4	4	4	4	4
5	0	0	5	5	0	5	5
6	0	0	6	6	6	0	6

Let $\mathcal{B}$ be a the subalgebra of $\mathcal{A}$ generated by the set $\{0, 1, 2\}$. This algebra was considered by V. L. Murskiĭ in [4]. The purpose of the paper is to prove the following

THEOREM. $card\{\mathbb{K} \mid HSP(\mathcal{A}) \supseteq \mathbb{K} \supseteq HSP(\mathcal{B})$ *and* $\mathbb{K} = HSP(\mathbb{K})\} = 2^{\aleph_0}$.

It is clear that the theorem implies existence of a subvariety of $HSP(\mathcal{A})$ which is not generated by any finite algebra. Moreover, by straightforward arguments we also see that $HSP(\mathcal{A})$ has infinitely many finite subdirectly irreducible algebras, as opposed to a congruence distributive variety generated by a finite set of finite similar algebras (see [2], Corollary 3,4).

A theorem similar to the above one has been proved by A. Wroński (unpublished). Precisely, by taking the seven-elements Lyndon's algebra (see [3]) as $\mathcal{B}$ and a certain ten-elements algebra as $\mathcal{A}$ A. Wroński proved that between varieties generated by $\mathcal{A}$ and $\mathcal{B}$, respectively, there are infinitely many varieties.

§3.

To prepare the proof of our theorem we need some technical lemmas. The first three of the lemmas may be proved without any difficulty.

LEMMA 2 (cf. [4]). *Let* α, β *be such that* $l(\alpha) = l(\beta)$. *Then,* $\alpha = \beta$ $Id(\beta)$ *iff* $V(\alpha) = V(\beta)$ *and* $C(\alpha) = C(\beta)$.

Notice that, by Lemmas 1 and 2, all identities $\alpha_n = \beta_n$ $(n \geq 2)$ are valid in $\mathcal{B}$.

LEMMA 3. *For any identity* $\alpha = \beta$ *valid in* $\mathcal{A}$*:*

(i) $V(\alpha) = V(\beta)$
(ii) if $\{x_i, x_i\} \notin C(\alpha)$ *for all* $i < \omega$, *then* $l(\alpha) = l(\beta)$ *and* $C(\alpha) = C(\beta)$.

LEMMA 4. *For any term* α *and valuation* v *in* $\mathcal{A}$*:*

(i) If $v(l(\alpha)) = 3$, *then* $v(\alpha) \in \{0, 3, 5\}$
(ii) if $v(l(\alpha)) = a \in \{2, 4, 5, 6\}$, *then* $v(\alpha) \in \{0, a\}$

LEMMA 5. *For any identity $\alpha = \beta$ from $Id(\mathcal{A})$, if $\alpha \in \mathbb{P}_n$ then $\beta \in \mathbb{P}_n$, for all $n \geq 2$.*

PROOF. Assume $\alpha = \beta \in Id(\mathcal{A})$ and $\alpha \in \mathbb{P}_n$. By Lemma 3 we infer that $\underline{4}eta \in \mathbb{K}_n$. Then, we must show that in β there is an occurrence of x_2 being simultaneously conjugate with some occurrence of x_1 and of x_3. Since α belongs to $\mathbb{P}_n$, there are subterms α_i $(i = 1, 2, 3)$ of the term α such that $l(\alpha_i) = x_i$, respectively, and there is a sequence (possibly empty) $\beta_1, \ldots, \beta_k$ of subterm of α such that one of the following terms is a subterm of α:

(i) $(\ldots((\alpha_2\alpha_1)\beta_1)\beta_2)\ldots\beta_k)\alpha_3$
(ii) $(\ldots((\alpha_2\alpha_3)\beta_1)\beta_2)\ldots\beta_k)\alpha_1$
(iii) $\alpha_1(\ldots(((\alpha_2\alpha_3)\beta_1)\beta_2)\ldots\beta_k)$
(iv) $\alpha_3(\ldots(((\alpha_2\alpha_1)\beta_1)\beta_2)\ldots\beta_k)$

CASE 1. A term of the form (i) is a subterm of α. Let v be a valuation in $\mathcal{A}$ such that $v(x) = 3$, when $x = x_2$, $v(x) = 4$, when $x = x_3$, and $v(x) = 2$, otherwise. For it, by Lemma 4, we obtain $v(\alpha) = 0$, and so, $v(\beta) = 0$. Hence, by the definition of the operation in $\mathcal{A}$ and valuation v, we conclude that there exists a subterm of β of the form $\gamma \circ \delta$ such that $v(\gamma) = 5$ and $v(\delta) = 4$. Since $v(\delta) = 4$ then, by the definition of v and Lemma 4, we get $l(\delta) = x_3$. Furthermore, $v(\gamma) = 5$ implies that γ is of the form $(\ldots((\gamma_1\gamma_2)\gamma_3)\ldots\gamma_{p-1})\gamma_p$, where $v(\gamma_1) = 3$ and $v(\gamma_2) = 2$, for some terms $\gamma_1, \ldots, \gamma_p$. In the way similar to the one followed before we settle that $l(\gamma_1) = x_2 \cdot \gamma_2$ is a subterm of β, so $C(\gamma_2) \subseteq c(\beta)$. This, together with $\beta \in \mathbb{K}_n$, allows us to conclude that $l(\gamma_2) = x_1$ or $l(\gamma_2) = x_3$. However, by Lemma 4 and the definition of v, we see that the case $l(\gamma_2) = x_3$ cannot take place. Finally, $\beta \in \mathbb{P}_n$.

CASE 2. A term of the form (ii) is a subterm of α. Take a valuation v in $\mathcal{A}$ defined as follows: $v(x) = 4$, when $x = x_1$, $v(x) = 3$, when $x = x_2$, and $v(x) = 2$, otherwise; and repeat the argumentation presented above.

CASE 3. A term of the form (iii) is a subterm of α. Let v be a valuation in $\mathcal{A}$ such that $v(x) = 6$, when $x = x_1$, $v(x) = 3$, when $x = x_2$, and $v(x) = 2$, otherwise. Using the same reasoning as in Case 1, we prove the existence of a subterm of β of the form $\delta(\ldots((\gamma_1\gamma_2)\gamma_3)\ldots\gamma_k)$, for some $\delta, \gamma_1, \gamma_2, \ldots, \gamma_k$, such that $v(\delta) = 6$, $v(\gamma_1) = 3$ and $v(\gamma_2) = 2$. Finally, by the definition of v

and Lemma 4, we obtain that $l(\delta) = x_1$, $l(\gamma_1) = x_2$ and $l(\gamma_2) = x_3$. Then, indeed, $\beta \in \mathbb{P}_n$.

Case 4. A term of the form (iv) is a subterm of α. Here it is enough to consider a valuation v in $\mathcal{A}$ such that $v(x) = 3$, when $x = x_2$, $v(x) = 6$, when $x = x_3$ and $v(x) = 2$, otherwise. Thus the proof of the Lemma is complete. Q.E.D.

Lemma 6. *Let $n \geq 2$ and $\alpha = \beta \in Id(\mathcal{A}) \cup \{\alpha_i = \beta_i \mid i \geq 2$ and $i \neq n\}$. Assume that $x_{i_1}, \ldots, x_{i_k}$ are all distinct variables occurring in α and β (see Lemma 3 and note to Lemma 2). Let α^+ and β^+ be the terms obtained from α and β, respectively, by simultaneous substitution of x_{i_j} by γ_j, where $\gamma_1, \ldots, \gamma_k$ are arbitrary terms. Moreover, let γ be a term containing α^+ as its subterm and let δ result from γ be replacing in γ some occurrence of α^+ by β^+. Then $\delta \in \mathbb{P}_n$ whenever $\gamma \in \mathbb{P}_n$.*

Proof. Let the above assumptions be satisfied. We have to consider two cases.

Case 1. $\alpha = \beta \in Id(\mathcal{A})$
The identity $\gamma = \delta$ is valid in $\mathcal{A}$, and so, by Lemma 5, the lemma is true.

Case 2. $\alpha = \beta \in \{\alpha_i = \beta_i \mid i \geq 2$ and $i \neq n\}$
Then the identity $\alpha = \beta$ is of the form $\alpha_i = \beta_i$, for some $i \neq n$. In the case when $i < n$ the proof of the lemma goes as the proof of the fundamental lemma in Murskiĭ [4]. Then, we may assume that $i > n$. The assumption $\gamma \in \mathbb{P}_n$ yields $V(\alpha^+) \subseteq \{x_1, \ldots, x_{a(n)}\}$, but $i > n$, so, $a(i) - a(i-1) > a(n)$ and, consequently, there must exist x_j such that $\{x_j, x_j\} \in C(\alpha^+)$ which is impossible, because α^+ is a subterm of γ. So, in this case the lemma is also true. Thus the proof is complete. Q.E.D.

Proof of the Theorem. Set $\Sigma_M = \{\alpha_i = \beta_i \mid i \in M\} \cup Id(\mathcal{A})$ for each $M \subseteq \omega \backslash \{0, 1\}$. Clearly, $HSP(\mathcal{A}) \supseteq (\Sigma_M)^* \supset HSP(\mathcal{B})$. Notice that Lemmas 1(ii) and 6 allow us to state that the identity $\alpha_n = \beta_n$ cannot be derived from the set $Id(\mathcal{A}) \cup \{\alpha_i = \beta_i \mid i \geq 2$ and $i \neq n\}$ and the tautological identities, $\alpha = \alpha$, be repeated applications of the rules of substitution and replacement of equals by equals. Therefore, making use of Parkhoff's theorem (see [1], th. 2, p. 170), we infer the existence of an algebra $\mathcal{C}_n \in HSP(\mathcal{A})$ such that $\alpha_n = \beta_n \notin Id(\mathcal{C}_n)$ and $\{\alpha_i = \beta_i \mid i \geq 2$ and $i \neq n\} \subseteq Id(\mathcal{C}_n)$. Hence $(\Sigma_M)^* \neq (\Sigma_M^+)^*$, whenever $M \neq M^+$. This completes the proof of our theorem. Q.E.D.

The question which immediately arises is whether there exist more natural examples of this kind. For instance, is there any finite algebra generating a congruence permutable (or modular) variety with $2^{\aleph_0}$ subvarieties? it is easy to observe that the algebra $\mathcal{A}$ is not of this kind.

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