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## COMMENTS ON THE LOGIC OF CONSTRUCTIBLE FALSITY (STRONG NEGATION)

Nelson has presented a constructive arithmetic with a negation operation ( $-$ ) different from the ordinary intuitionistic one ( $\neg$ ). In [5] he presents a variant of Kleene's realization semantics for intuitionistic arithmetic, and proves that relative to this interpretation the arithmetic language with  $-$  has the same expressive power as the usual intuitionistic one, and fact certain theories of arithmetic incorporating his negation are equivalent to corresponding systems of intuitionistic arithmetic. A Fitch style natural deduction formulation (cf.[1]) of the pure first order logic that may be extracted, from Nelson's work may be obtained from that for intuitionistic first order logic by first dropping the intuitionistic rule of negation introduction and then adding rules guaranteeing the De Morgan equivalences, their quantifier analogues, and the equivalence of anegated implication with the conjunction of its antecedent and the negation of its consequent. Given the De Morgan equivalences, conjunction and the existential quantifier may be treated as defined operators.

Thomason, in [8], develops a classical model theory for what he claims is the first order logic of [5]. It is similar to his and Kripke's model theory for intuitionistic predicate logic (cf. [4], [7]), with appropriate changes in the treatment of negation, but holding the domain of objects "existing" at the various "stages" constant. Now, Görnemann (in [2]) and others (cf. references in [2]) have shown that holding the domain constant in the model theory of intuitionistic logic characterizes the predicate logic obtained from intuitionistic logic by adding the intuitionistically invalid rule  $\underline{D}$ : from  $\forall x(Fx \vee A)$ , where  $x$  has no free occurrence in  $A$ , to infer  $(\forall x(Fx) \vee A)$ . It is thus no surprise that the logic of [8] is stronger than the one I have described, and validates  $\underline{D}$  (it may in fact be obtained from the

logic I describe by adding the rule  $\underline{D}$ .  $\underline{D}$ , however, is just as unacceptable from Nelson's point of view as it is from that of the intuitionists. Indeed, given the constructive derivability of excluded middle for atomic (and other decidable) formulas of arithmetic, the addition of  $\underline{D}$  to either intuitionistic or Nelson arithmetic, the would cause it to collapse into classical arithmetic.

Görnemann's result suggests the conjecture that a classical model theory for the logic I have described may be obtained by allowing the domain to "grow with time". This is in fact true. We may define a *Nelson model structure* as a triple  $(K, R, D)$ , where  $K$  is a non-empty set of "stages of investigation",  $R$  is the reflexive and transitive relation of "temporal succession" over  $K$ , and  $D$  a function assigning to each  $k$  in  $K$  a non-empty domain of objects "existing" at it, subject to the constraint that if  $hRk$ , then  $D(h)$  is a subset of  $D(k)$ . An *interpretation*  $I$  of a first order language on a model structure  $(K, R, S)$  is a function e.t.

i) for each term  $t$  of the language,  $I(t)$  is an object in the union, for  $k$  in  $K$ , of the  $D(k)$ , and

ii) for each  $n$ -adic predicate  $P$  and each  $k$  in  $K$ ,  $I(P, k)$  is a partial function from the  $n$ -th cartesian power of  $D(k)$  into the two-membered set  $\{T, F\}$ , with the restriction that if  $kRh$ , then  $I(P, k)$  is a subset of  $I(P, h)$ .

Define an interpretation  $I$  to *hit* a stage  $k$  iff  $I(t)$  is in  $D(k)$  for every term  $t$ ; where  $I$  is an interpretation on  $(K, R, D)$ ,  $x$  an individual variable,  $k$  an element of  $K$ , and  $d$  an object in  $D(k)$ , let  $I(d/x)$  be the interpretation on  $(K, R, D)$  differing from  $I$  at most in that  $I(d/x)(x) = d$ .

Where  $k$  is a stage of  $(K, R, D)$  and  $I$  an interpretation hitting  $k$ , the truth value  $V(I, k, A)$ , if any, given a formula  $A$  by  $I$  at  $k$  is defined by induction on the complexity of  $A$ :

- i)  $V(I, k, P(t_1, \dots, t_n)) = T$  iff  $I(P, k)(I(t_1), \dots, I(t_n)) = T$ ,  
 $V(I, k, P(t_1, \dots, t_n)) = F$  iff  $I(P, k)(I(t_1), \dots, I(t_n)) = F$ ;
- ii)  $V(I, k, (B \vee C)) = T$  iff  $V(I, k, B) = T$  or  $V(I, k, C) = T$ ,  
 $V(I, k, (B \vee C)) = F$  iff  $V(I, k, B) = F$  and  $V(I, k, C) = F$ ;
- iii)  $V(I, k, \neg B) = T$  iff  $V(I, k, B) = F$ ,  
 $V(I, k, \neg B) = F$  iff  $V(I, k, B) = T$ ;
- iv)  $V(I, k, (B \supset C)) = T$  iff for every  $h$  such that  $kRh$ , if  
 $V(I, h, B) = T$ , then  $V(I, h, C) = T$ ,
- v)  $V(I, k, \forall x B) = T$  iff for every  $h$  such that  $kRh$  and every  $d$   
in  $D(h)$ ,  $V(I, d/x, h, B) = T$ , and  
 $V(I, k, \forall x B) = F$  iff for some  $e$  in  $D(k)$ ,  $V(I(e/x), k, B) = F$ .

*Validity* for formulas (argument forms) is truth (truth preservation) at every stage of every Nelson model structure on every interpretation hitting that stage.

Gurevich (in [3]) has proven the (weak, but strong completeness could be proven in the same way) completeness of essentially the logic described here relative to essentially this model theory. His proof proceeds by showing that a predicate logic incorporating both  $-$  and  $\neg$  and relating them by the rule that  $-A$  implies  $\neg A$  is a conservative extension of intuitionistic logic (this result is implicit in Nelson's paper), and noting that a model, in the sense of [4] and [7], for intuitionistic logic can always be extended to a model of the combined logic treating  $-$  in essentially our way.

Examination of his proof shows that it would still go through if the combined logic were strengthened by postulating the *equivalence* of  $-A$  and  $\neg A$  for atomic  $A$ . (On Gurevich's construction an atomic formula is Nelson-false at a subset of the stages at which it is intuitionistically false. To get the stronger result we merely make it Nelson-false at all the stages at which it is intuitionistically false). Given the decidability of the atomic formulas of arithmetic, this provides a model-theoretic proof of the equivalence of appropriate systems of intuitionistic and Nelson arithmetic, which may be of interest to those who find classical model theory more interesting than realization semantics.

The strong completeness of the logic described relative to the model theory described may also be proven by a Henkin style proof very similar to Thomason's. Letting a *saturated* set of formulas of a first order language be one that is consistent, deductively closed, contains at least one disjunct of each of its disjunctions, and contains at least one instance of each of its disjunctions, and contains at least one instance of each of its existential quantifications, there are two lemmas asserting the existence of appropriate saturated sets, which we state for the case of denumerable languages. Their proof is straightforward. First, where  $G$  is a set of formulas and  $E$  a formula of the same language not derivable from it, there is, for a language formed from the first by the addition of a denumerable set of terms, a saturated set  $G^*$  containing  $G$  and such that  $E$  is not derivable from  $G^*$  (this is a form of Lindenbaum lemma). Second, in order to ensure the existence of the "later" stage with a larger domain needed to prevent the verification of undesirable quantified formulas, if  $G$  is a saturated set and  $\forall x A$  a formula of the language not in  $G$ , there is, in a language with denumerably many

extra terms, a term  $t$  and a saturated set  $G^{**}$  such that  $G^{**}$  contains  $G$  but does not contain  $A(t/x)$ .

It seems not unlikely that Thomason was led to his model theory by a consideration of the system of Fitch [1]. This system is very different in its motivation from Nelson's (see, in this connection, [0]), but a first order logic somewhat similar to the one described may be extracted from it in which the rule  $\underline{D}$  is accepted (cf. also [6]).

## References

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