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VERBAL COPIES

The aim of this note is to analyse the problems connected with constraints on descriptions (called verbal copies) imposed by limitations of knowledge and/or observability, and constraints imposed by the language used.

Let X a nonempty set, whose elements will be interpreted as elementary features, or elementary descriptors, and let $\equiv$ be an equivalence relation in X, partitioning it into equivalence classes $X_1, \ldots, X_N$.

We shall denote by S the class o all vectors $\underline{A} = (A_1, \ldots, A_N)$ with $A_i \subset X_i$, $i = 1, \ldots, N$. In the sequel, elements of S will be denoted by capital letters $\underline{A}, \underline{B}, \ldots$ and their coordinates will be denoted by the same letter with an index, so that B_i is the i-th coordinate of $\underline{B} = (B_1, \ldots, B_N)$, etc.

Given $\underline{A}, \underline{B} \in S$, define $\underline{A} \cdot \underline{B} = (A_1 \cap B_1, \ldots, A_N \cap B_N)$, and $\underline{A} + \underline{B} = (A_1 \cup B_1, \ldots, A_N \cup B_N)$. If $\underline{A} \cdot \underline{B} = \underline{A}$, we write $\underline{A} \subset \underline{B}$.

These operations satisfy the usual laws of idempotence, commutativity, distributivity, etc., e.g. $\underline{A} + \underline{A} = \underline{A}$, $\underline{A} + (\underline{B} \cdot \underline{C}) = (\underline{A} + \underline{B}) \cdot (\underline{A} + \underline{C})$, and so forth.

Let now Z be the set of objects to be described. With each $z \in Z$ we associate a subset S_z of S, satisfying the following conditions (A1)-(A3).

(A1) If $\underline{A}, \underline{B} \in S_z$, then $\underline{A} \cdot \underline{B} \in S_z$.
(A2) $\underline{A} \in Sz, A_i = \emptyset \Rightarrow (\forall \underline{B})[\underline{B} \in S_z \Rightarrow B_i = \emptyset]$.

Before formulating (A3), let us record the following

Proposition 1. *If $\underline{A}, \underline{B} \in S_z$ and $A_i \neq \emptyset$, then $A_i \cap B_i \neq \emptyset$.*

Indeed, under the assumptions, we have $B_i \neq \emptyset$ by (A2); since $\underline{A}\cdot\underline{B} \in S_z$ by (A1), we must have $A_i \cap B_i \neq \emptyset$ by (A2).

Extending Proposition 1 to more than two factors, we can define $V_i^z = \bigcap_{\underline{A}\in S_z} A_i$, and we have

PROPOSITION 2. $\underline{V}^z = (V_1^z, \ldots, V_N^z) \in S_z$.

Let $I = \{1, \ldots, N\}$ and

$$I(z) = \{i \in I : (\exists \underline{A})\underline{A} \in S_z, A_i \neq \emptyset\}.$$

$I(z)$ will be called the *base* of z. We have then, for $\underline{V}^z$ from Proposition 2:

PROPOSITION 3. *$V_i^z \neq \emptyset$ iff $i \in I(z)$.*

Indeed, if $i \notin I(z)$, when $A_i \neq \emptyset$ for any $\underline{A} \in S_z$, hence $V_i^z = \emptyset$. If $i \in I(z)$, all sets A_i for $\underline{A} \in S_z$ are nonempty, and moreover, by extension of Proposition 1, their product is also nonempty.

Given $i \in I(z)$ and $A \subset X_i$ define

$$\underline{Q} = \underline{Q}^z(A) = (Q_1, \ldots, A_N)$$

where

$$Q_j = \begin{cases} A & \text{for} \quad j = i, \\ X_j & \text{for} \quad j \in I(z), i \neq i, \\ \emptyset & \text{for} \quad j \notin I(z). \end{cases}$$

We can now formulate

(A3) $\underline{A} \in S_z, A_i \neq \emptyset \Longrightarrow \underline{Q}^z(A_i) \in S_z$.

This means that if one takes an element from S_z and extends maximally all nonempty coordinates except one, one still gets an element of S_z.

The elements of S_z are to be interpreted as abstractions from descriptions of the objects z, the vector $\underline{A} = (A_1, \ldots, A_N)$ signifying the conjunction like "a is A_1 and ... and z is A_N". The empty sets correspond to sets of features which are not applicable to z. The maximal elements $\underline{Q}^z(A)$ are the descriptions concerning one feature only, while $\underline{V}^z$ is the most exact description available.

We shall now describe the constraints on S_z imposed by (a) knowledge and observability limitations, and (b) language used in descriptions.

Let Γ be a family of sets defined as

$$G \in \Gamma \text{ iff } G \neq \emptyset \text{ and } (\exists i) G \subset X_i.$$

Clearly, the index i in the last definition is determined uniquely, in view of the fact that the sets X_i are disjoint, so that to each $G \in \Gamma$ one can assign $i_G \in I$.

Next, let $G_1, \ldots, G_r$ be elements of Γ such that the indices i_{G_k}, $k = 1, \ldots, r$ are all distinct and belong to $I(z)$. Denote

(∗) $\underline{C}^z(G_1, \ldots, G_r) = \underline{Q}^z(G_1) \cdot \ldots \cdot \underline{Q}^z(G_r)$.

We have

PROPOSITION 4. $\{j : C_j^z(G_1, \ldots, G_r) \neq \emptyset\} = I(z)$.

Indeed, $C_j^z(G_1, \ldots, G_r) = Q_j^z(G_1) \cap \ldots \cap Q_j^z(G_r)$, and the last product is empty for $j \notin I(z)$, and equals either X_j or G_k with $i_{G_k} = j$ otherwise. The assertion follows because $G_k \neq \emptyset$ by assumption.

The knowledge (about z), or the effect of restrictions on observability will be represented as a set K of vectors of the form (∗) such that $K \subset S_z$. A vector $\underline{C}^z(G_1, \ldots, G_r)$ in K will be interpreted as representing the assertion that z is such that it has features from G_k on attribute i_{G_k} for $k = 1, \ldots, r$.

It will be assumed that K is closed under the operation of product, i.e.

$$\underline{A}, \underline{B} \in K \Rightarrow \underline{A} \cdot \underline{B} \in K.$$

As in the case of S_z, one can form the vector $\underline{W}^z = (W_1^z, \ldots, W_N^z)$ with $W_i^z = \bigcap_{\underline{A} \in K} A_i$. As an obvious consequence of inclusion $K \subset S_z$ we have

PROPOSITION 5. $\underline{V}^z \subset \underline{W}^z$.

The knowledge and/or observability is said to be *full* (resp. full on i-th attribute) if $\underline{V}^z = \underline{W}^z$ (resp. $V_i^z = W_i^z$).

To formalize the linguistic constraints, assume further that we have a set D (vocabulary), and a mapping

$$f : \Gamma \to D \cup \{\oplus\}$$

where $\oplus \notin D$ is a special symbol signifying "no name". If $f(G) = d \in D$, we say that d is the name, or linguistic representation, of the set G of features (e.g. "shorter than 5 cm", etc.). If $f(G) = \oplus$, there is no special term for the features in the set G in the considered language.

Thus, $\Gamma^* = f^{-1}(D)$ is the subset of Γ of those sets of features which have names in D. A name need not be a single term: it may be composed of other names.

Given S_z, let S^*_z be the subset of S_z consisting of all those vectors whose coordinates are all in Γ^*.

A *verbal copy* of z may be identified with a string $v = d_1 d_2 \dots d_n$ of elements of D ("z is d_1 and ... and z is d_n").

Consequently, to a verbal copy v one may assign a string of sets $f^{-1}(d_1), f^{-1}(d_2), \dots, f^{-1}(d_n)$, and a class $U(v)$ of strings $U_1, \dots, U_n$ of elements of Γ^*, where $U_i \in F^{-1}(d_i)$ for $i = 1, \dots, n$, to be called the raw interpretation of v. If $f^{-1}(d_i)$ consists of m_i elements, the number of raw interpretations in $m_1, \dots, m_n$.

Given a raw interpretation $U_1, \dots, U_n$, one can form a reduced interpretation $G_1, \dots, G_N$, where $G_i = \bigcap^{(i)} U_j$, the upper index (i) signifying that the intersection is extended over those U_j which are contained in X_i. Thus, a reduced interpretation is an element of S.

Let $A(v)$ be the class of all reduced interpretations of the verbal copy v. If $A(v)$ contains more than one element, v is said to be ambiguous,otherwise it is unequivocal. If $A(v) \subset S_z$, the verbal copy is said to be faithful, and if $\underline{V}^z \in A(z)$, it is said to be exact (observe that an exact verbal copy may be ambiguous: it may happen that there are more interpretations, of which only one is equal $\underline{V}^z$).

The problem arises how rich the vocabulary D must be in order for exact verbal copies to exist. We have here

Proposition 6. *For the existence of exact verbal copies it is sufficient that*

$$(**) \quad (\forall i)(\forall x \in x_i)(\exists d_1, \dots, d_r \in D) : \bigcap_{j=1}^{r} f^{-1}(d_j) = \{x\}.$$

Indeed, this condition asserts that any single feature x is expressible, in effect, as a conjunction of terms from D. If $\underline{V}^z$ contains components which are not singletons, they may be represented as alternatives.

If $(**)$ holds, one may associate with every x the minimal number r of descoriptors $d_1, \dots, d_r$ needed to identity x, say $m(x)$. Let $M_i = sup_{x \in X_i} m(x)$. If X_i contains r_i elements, then $M_i = min\{k : 2^k \geq r_i\}$, since specification of each d_i provides binary information (whether $x \subseteq f^{-1}(d_i)$ or not).

The formalism introduced here for the description of objects z (or:

"states of the world") is particulary convenient (preferable to a representation in terms of relational systems) for the analysis of the dynamic aspects, namely when one assumes that the sets S_z change in time. The description of change is then reduced to a convenient representation in terms of a vector of functions $\underline{V}^z(t) = (V_1^z(t), \ldots, V_N^z(t))$, for which there exist standard tools of analysis.

Another extension is obtained when the relation between elements of the vocabulary D and sets of features is fuzzy, i.e. given in terms not of a function f, but in terms of a membership function in a suitable fuzzy set.

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