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STRUCTURE OF CONCEPTS

The purpose of this note is to analyse the structure of concepts, treated as fuzzy subsets of a certain set. The concepts are assumed to be composed out of more elementary ones. Such an approach was developed mainly to get an empirical access to the values of membership functions of concepts.

Let X be a nonempty set, and let $\underline{F}$ be the class of all functions $f : X \to [0,1]$. Each $f \in \underline{F}$ represents then a fuzzy subset of X.

For $f, g \in \underline{F}$, let $d_1(f,g) = sup_{x\in X} \mid f(x) - g(x) \mid$ and $d_2(f,g) = \int (f(x) - g(x))^2 p(x)dx$, where p is a certain probability distribution on X. Then d_1 and d_2 are two matrices on $\underline{F}$.

Putting $h_i(f,g) = 1 - d_i(f,g)$, $i = 1, 2$, we establish a function from $\underline{F} \times \underline{F}$ to $[0,1]$, hence a fuzzy relation in $\underline{F}$, that of a fuzzy identity of fuzzy subsets of X.

Let now L be a set, whose elements will be interpreted as labels, or names, of concepts. We assume that for elements of L one can speak of their negation, conjunction and disjunction, denoted respectively by $\sim, \wedge$ and $\vee$. The set L need not, however, be closed under these operations, i.e. the conditions $u, v \in L$ do not, in general, entail $\sim u \in L$, $u \wedge v \in L$ or $u \vee v \in L$. This means that a concept being a conjunction of two concepts need not have its own name, etc.

Next, let $T : \underline{F} \to L \cup \{\oplus\}$, where $\oplus \notin L$ is a special symbol signifying "no name". The relation $T(f) = \oplus$ means that the fuzzy set with membership function f has no name, simple or composite, in the set L.

It will be assumed that the function T satisfies the following conditions:

(L1) $(\forall u \in L)(\exists f \in \underline{F}) : T(f) = u$.
(L2) If $T(f) = u$ and $\sim u \in L$, then $T(1 - f) =\sim u$.

(L3) If $T(f_i) = u_i, i = 1, \ldots, n$, and $u_1 \wedge \ldots \wedge u_n \in L$, then $T(min(f_1, \ldots, f_n)) = u_1 \wedge \ldots \wedge u_n$.

(L4) If $T(f_i) = u_i, i = 1, \ldots, n$, and $u_1 \vee \ldots \vee u_n \in L$, then $T(max(f_1, \ldots, f_n)) = u_1 \vee \ldots \vee u_n$.

The condition L1 asserts that each concept with label from L is representable by at least one fuzzy set in X, while conditions L2-L4 assert a certain consistency between operations on labels in L and set-theoretical operations on fuzzy sets.

Given a set $A \subset \underline{F}$, its diameter is defined as $D_i(A) = sup_{g,g \in A} d_i(f, g)$. The diameter depends on the choice of the metric d_i.

We have

Proposition 1. *Suppose that* $u, \sim u \in L$. *Then for* $i = 1, 2$

$$D_i(T^{-1}(u)) = D_i(T^{-1}(\sim u)).$$

Proof. Observe first that both the sets $T^{-1}(u)$ and $T^{-1}(\sim u)$ are well defined. We have $f \in T^{-1}(u)$ iff $1 - f \in T^{-1}(\sim u)$ by L2. The assertion follows now from the fact that either of the metrics d_1 and d_2 is invariant under the change of sign and addition of a constant.

If $u \wedge v \in L$, define

$$T^{-1}_{min}(u \wedge v) = \{h : h = min(f, g), f \in T^{-1}(u), g \in T^{-1}(v)\}.$$

Similarly, if $u \vee v \in L$, let

$$T^{-1}_{max}(u \vee v) = \{h : h = max(f, g), f \in T^{-1}(u), g \in T^{-1}(v)\}.$$

By L3 and L4 that we have

Proposition 2.

$$T^{-1}_{max}(u \wedge v) \subset T^{-1}(u \wedge v) \text{ and } T^{-1}_{max}(u \vee v) \subset T^{-1}(u \vee v).$$

Next, for the conjunction and disjunction of concepts, for the case of metric d_1, we have

Proposition 3. *Let* $u, v, u \wedge v \in L$. *Then*

$$D_1(T^{-1}_{min}(u \wedge v)) \leq max[D_1(T^{-1}(u)), D_1(T^{-1}(v))].$$

Proposition 4. *Let* $u, v, u \vee v \in L$. *Then*

$$D_1(T^{-1}_{max}(u \vee v)) \leq max[D_1(T^{-1}(u)), D_1(T^{-1}(v))].$$

Proof. We shall prove Proposition 3, the proof of Proposition 4 being analogous. Observe that $T^{-1}_{min}(u \wedge v)$ is well defined in view of the assumption $u \wedge v \in L$. By L3 we have

$$f \in T^{-1}(u), g \in T^{-1}(v) \Rightarrow min(f, g) \in T^{-1}_{min}(u \wedge v).$$

Let $D_1(T^{-1}(u)) = a$, $D_1(T^{-1}(v)) = b$. Then for all $f, f' \in T^{-1}(u)$ and all $x \in X$ we have $f'(x) - a \leq f(x) \leq f'(x) + a$, and similarly, $g'(x) - b \leq g(x) \leq g'(x) + b$ for all $x \in X$ and $g, g' \in T^{-1}(v)$. We can therefore write

$$\begin{aligned} min(f(x), g(x)) &\leq min(f'(x) + a, g'(x) + b) \leq \\ &\leq min(f'(x), g'(x)) + max(a, b). \end{aligned}$$

In a similar way we get

$$min(f(x), g(x)) \geq min(f'(x), g'(x)) - max(a, b),$$

which yields $| \; min(f(x), g(x)) - min(f'(x), g'(x)) \; | \leq max(a, b)$. Taking supremum first with respect to $x \in X$, and then with respect to f, f', g, g' we obtain the assertion.

Let now $u \in L$ and consider all possible decompositions $u = u_1 \wedge \ldots \wedge u_r$, with $u_i \in L$, $i = 1, \ldots, r$. If for any k $(2 \leq k \leq r)$ and any $1 \leq i_1 < \ldots < i_k \leq r$ we have $u_{i_1} \wedge \ldots \wedge u_{i_k} \notin L$, we say that $(u_1, \ldots, u_r)$ constitutes a basic decomposition of u. Obviously, basic decompositions need not be unique.

The components which appear in every basic decomposition of u are called the core of the concept u.

Decomposing basic components $u_1, u_2, \ldots$, one obtains further decomposition, ultimately reducible to a tree, with atoms at the top (v is called an atom in L, if it cannot be represented as a conjunction of any concepts in L different from itself).

For $u \in L$, let

$$I(u) = \{x \in X : (\forall f \in \underline{F}) f \in T^{-1}(u) \Rightarrow f(x) = 1\}.$$

The set $I(u)$ will be called the set of ideal objects for the concept u. Similarly, let

$$E(u) = \{x \in X : (\exists f \in \underline{F}) f \in T^{-1}(u) \text{ and } f(x) = 1\}.$$

The set $E(u)$ will be called the set of exemplars of u. We have

PROPOSITION 5. *Let $(u_1, \ldots, u_r)$ be a basic decomposition of u, and let $x \in I(u)$. Then*

$$(\forall i = 1, \ldots, r)(\forall f \in \underline{F}) f \in T^{-1}(u) \Rightarrow f(x) = 1.$$

PROOF. Let $(u_1, \ldots, u_r)$ be a basic decomposition of u, and let $f_i \in T^{-1}(u_i)$, $i = 1, \ldots, r$. By L3 we have $f = min(f_1, \ldots, f_r) \in T^{-1}(u)$. If $x \in I(u)$, then $f(x) = 1$, hence $f_i(x) = 1$ for all i.

PROPOSITION 6. *If $x \in E(u)$, and u_i belongs to the core of u, then $f_i(x) = 1$ for $f_i \in T^{-1}(u_i)$.*

PROOF. Let $x \in E(u)$ and let f be the function in $T^{-1}(u)$ with $f(x) = 1$. If u_i is in the core of u, then any decomposition of u into $u_1 \wedge \ldots \wedge u_n$ must contain u_i. By L3 we have $f = min(f_1, \ldots, f_n)$, hence $f_i \geq f$, and we must have $f_i(x) = 1$.

Thus, any ideal of u must belong to any set in the basic decomposition in full degree; any exemplar of u must belong in full degree to any set in the core of u.

When a concept u is reduced to atomic concepts $v_1, \ldots, v_N$ it is sometimes (e.g. in the case of some psychological concepts) possible to obtain a classification of an object $x \in X$ into classes defined in terms of $v_1, \ldots, v_N$ and consequently, build an estimator of $f(x)$ for $f \in T^{-1}(u)$. The general idea is based on the following approach to the notion of classification. By a classification of X one understands a mapping $t : X \to \underline{C} = \{C_1, C_2, \ldots\}$, where C_i's are categories (defined through $v_1, \ldots, v_N$). The function t may be partial and it may change between classifying subjects, and occasionally for the same subject. The relation $t(x) = C_j$ means that object x has been classified into category C_j (in particular, $t(x)$ may be an assertion about x, using fuzzy concepts, and with a modal frame, e.g. "I am certain that", etc.).

Classification of type 1 is such that in addition one has a function $\varphi : X \to \underline{C}$, where $\varphi(x)$ is the true category of object x. Then the measure of quality of classification (with respect to x) is $P(t(x) = \varphi(x))$.

In classification of type 2, the notion of true category makes no sense (e.g. in the case of grading student's exams). In such a case , the quality of classification (with respect to x) is expressed by the requirement that $P(t(x) = t'(x))$ should be as high as possible for as many $x \subseteq X$ as possible, where t and t' are two independent classifications. Now, $P(t(x) = t'(x)) =$

$\sum_j P(t(x) = C_j)P(t'(x) = C_j)$, which is reducible to $\sum_j P(t(x) = C_j)^2$ in the case of identically distributed t and t'.

The empirical access to the values of membership function $f(x)$ is provided, in part, by plausible assumptions, stating that $P(t(x) = C_j)$ depends on the values of membership functions which enter the definition of the category C_j. The details of this construction may be found in [1].

References

[1] M. Nowakowska, *Methodological problems of measurement of fuzzy concepts in the social sciences*, **Behavioral Science** 22 (1977), pp. 107–115.

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