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R_+ IS CONTAINED IN T_+

Although the system T of ticket entailment is obviously related to its cousins E and R (it is, after all, a relevant logic, and moreover a subsystem of E and therefore of R), it is motivated along quite distinctive lines in Anderson and Belnap [1975]. It would seem, accordingly, that T is more nearly akin to the system $P - W$ studied in Martin [1978] (see also Anderson and Belnap [1975], pp. 94–95, noting that $T \to -W$ is $P - W$) than to E and R. The result presented here, however, at least suggests the contrary.

As the title announces, a formula is provable in R_+ only if (in fact, iff) its "translation" is provable in T_+. Below we merely state this result precisely and give the major lemmas for its proof, sparing the reader the details, which are at any rate straightforward.

We assume R_+ and T_+ to be formulated axiomatically as in Anderson and Belnap [1975]. T_+^t is obtained from T_+ by first adding the propositional constant t to the vocabulary, and then adding the following *rules* to T_+:

$$\text{(a)}\ \frac{\vdash A}{\vdash t \to A} \qquad \text{(b)}\ \frac{\vdash t \to A}{\vdash A}$$

For each $wff A$ of T_+^t, define A' as $A[t/p_i]$ (the proper substitution of p_i for t in A), where p_i is the first propositional variable (in some assumed ordering thereof) which does not occur in A. And define t_A as $(p_i \& (p_i \to (p_i \to p_i)) \& (p_i \to (p_h \to p_h)) \& \ldots \& (p_i \to (p_k \to p_k))$, where p_i is as before, and $p_h, \ldots, p_k$ are the propositional variables occurring in A. Then

Lemma 1. $\vdash_{T_+} tA$ *iff* $\vdash_{T_+} (t'_{A'} \to A')$.

We have noted (as Anderson and Belnap [1975] essentially records, pp. 100–101) that if a theory of modality is wanted for T, the definition scheme

$$\Box A = (t \to A)$$

suggests itself. On this scheme, a slight modification of the coding of R_+ into E_+ in Meyer [1966] (itself and adaptation of the well-known Mckinsey-Tarski 1948 coding of intuitionist logic into $S4$) codes R_+ into T_+ as well. Specifically, let A, B, C be arbitrary $wffs$ of R_+. Define A^* inductively as follows:

$A^* = (\Box A)$, for A a propositional variable;
$(B\&C)^* = (B^*\&C^*)$;
$(B \vee C)^* = (\Box(B^* \vee C^*))$;
$(B \to C)^* = (B^* \to C^*)$.

Then

LEMMA 2. $\vdash_{R_+} A$ *iff* $\vdash_{T_+} tA^*$.

With A^*, A' and $t_{A'}$ defined as above, it follows immediately from Lemmas 1 and 2 that

THEOREM. $\vdash_{R_+} A$ *iff* $\vdash_{T_+} (t_{A^{*\prime}} \to A^{*\prime})$.

Before closing, we note as a corollary of this theorem: If R_+ is undecidable, T_+ is likewise. So a negative answer to the R_+ decision question would settle negatively all of the still-open decision questions for the major relevant systems T, E and R.

References

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