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BOCHVAR'S ALGEBRAS AND CORRESPONDING PROPOSITIONAL CALCULI

This is an abstract of the paper which is to appear in “Issledovanie po neklassicheskim logikam i teorii mnozhestv” (“Nauka”).

In [1] D. A. Bochvar formulated a 3-valued logic. He analysed the paradoxes of Russel and Weyl, and by means of the logic he proved that the paradox formulae were meaningless.

In this paper the class of algebras (B_n -algebras) corresponding to n -valued generalizations of the Bochvar's 3-valued logic is investigated. The class is defined axiomatically. The axiomatization for Bochvar's n -valued logic B_n is obtained on the basis of algebraic axiomatization.

1.

A B_n -algebra ($2 < n < \aleph_0$) is a universal algebra $\mathcal{A} = \langle A, \cup, \cap, \sim, J_0, \dots, J_{n-1}, 0, 1 \rangle$, where A is a nonempty set of elements, 0 and 1 are constant elements of A , \cup and \cap are binary operations on elements of A , and $\sim, J_0, \dots, J_{n-1}$ are unary operations on elements of A obeying the following axioms:

- A1. $x \cup x = x$
- A2. $x \cup y = y \cup x$
- A3. $x \cup (y \cup z) = (x \cup y) \cup z$
- A4. $x \cap (y \cup z) = (x \cap z) \cup (x \cap y)$
- A5. $\sim \sim x = x$
- A6. $\sim 1 = 0$

- A7. $\sim(x \cup y) = \sim x \cap \sim y$
A8. $0 \cup x = x$
A9. $J_{n-1}J_ix = J_ix, \quad 0 \leq i \leq n-1$
A10. $J_0J_ix = \sim J_ix, \quad 0 \leq i \leq n-1$
A11. $J_iJ_jx = 0, \quad 0 < i < n-1, \quad 0 \leq j \leq n-1$
A12. $J_i(\sim x) = J_{n-1-i}x$
A13. $J_ix = \sim(J_0x \cup \dots \cup J_{i-1}x \cup J_{i+1}x \cup \dots \cup J_{n-1}x)$
A14. $J_ix \cup \sim J_ix = 1, \quad 0 \leq i \leq n-1$
A15. $(J_ix \cup J_kx) \cap J_ix = J_ix, \quad 0 \leq i, k \leq n-1$
A16. $x \cup Jix = x, \quad n-1 \geq i \geq n-1-i$
A17. $J_k(x \cup y) = \bigcup_{j=0}^k (J_kx \cap J_jy) \cup \bigcup_{i=0}^k (J_ix \cap J_ky) \quad 0 \leq k < [\frac{n}{2}]$
A18. $J_k(x \cup y) = \bigcup_{i=1}^{n-1} (J_ix \cap J_ky) \cup \bigcup_{i=k}^{n-1} (J_kx \cap J_iy) \cup$
 $\bigcup_{i=k}^{n-1} (J_ix \cap J_k \sim y) \cup \bigcup_{i=k}^{n-1} (J_kx \cap J_i \sim y) \cup$
 $\bigcup_{i=k}^{n-1} (J_i \sim x \cap J_ky) \cup \bigcup_{i=k}^{n-1} (J_k \sim x \cap J_iy), \quad n-1 \geq k \geq [\frac{n+1}{2}]$
A19. $(\forall i)(0 \leq i \leq n-1)J_ix = J_iy \Rightarrow x = y$
 $[m]$ is the largest integer k such that $k \leq m$.

B_n -algebras are quasi-lattices in the sense of Plonka [2] with the operation of involution \sim for which De Morgan axioms hold, and with unary J -operations J_0, \dots, J_{n-1} .

The algebra $\mathbb{B} = \langle R_n, \cup, \cap, \sim, J_0, \dots, J_{n-1}, 0, 1 \rangle$, where $R_n = \{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$, $\sim x = 1 - x$, $x \cup y = \min(\max(x, y), \max(\sim x, x), \max(\sim y, y))$, $x \cap y = \max(\min(x, y), \min(\sim x, x), \min(\sim y, y))$, $J_ix = \begin{cases} 1, & x = \frac{1}{n-1} \\ 0, & x \neq \frac{1}{n-1} \end{cases}$, $0 \leq i \leq n-1$, is an example of the B_n -algebra.

PROPOSITION. *The class of all B_n -algebras is a quasi-variety but it is not a variety.*

$$x \leq y \quad \text{iff} \quad \begin{aligned} J_i(x \cap y) &= J_ix & [\frac{n+1}{2}] \leq i \leq n-1, \\ J_j(x \cup y) &= J_jy & 0 \leq j < [\frac{n}{2}]. \end{aligned}$$

THEOREM 1.1. *The relation \leq is a partially ordered relation on A .*

A subset F of the set A is a filter of the B_n -algebra \mathcal{A} iff (i) $1 \in F$, (ii) if $x, y \in F$ then $x \cap y \in F$, (iii) if $x \in F$ and $x \leq y$ then $y \in F$, (iv) if $x \in F$ then $J_{n-1}x \in F$.

THEOREM 1.2. *If F is a filter, then the relation R on A defined by xRy iff $\sim J_i x \cup J_i y, J_i x \cup \sim J_i y \in F$ $[\frac{n+1}{2}] \leq 1 \leq n-1, \sim J_j x \cup J_j y, J_j x \cup \sim J_j y \in F, 0 \leq j < [\frac{n}{2}]$, is a congruence relation.*

Let us consider the algebra $\mathbb{B}_m = \langle R_m, \cup, \cap, \sim, J_0, \dots, J_{n-1}, 0, 1 \rangle$. Let f be a mapping of the set $\{0, 1, \dots, m-1\}$ into the set $\{0, 1, \dots, n-1\}$ ($m < n$) such that (1) $f(0) = 0$, (2) $f(m-1) = n-1$, (3) $\forall x, y \in \{0, \dots, m-1\} x \leq y = f(x) \leq f(y)$, (4) $f(m-1-i) = n-1-f(i)$ where $0 \leq i \leq m-1$. From the definition of f it follows that such f does not exist if m is odd and n is odd and n is even. The algebra

$\mathbb{B}_m^f = \langle R_m, \cup, \cap, \sim, J_0, \dots, J_{f(1)}, \dots, J_{f(2)}, \dots, J_{f(m-2)}, \dots, J_{n-1}, 0, 1 \rangle$ where $J_{f(i)} = J_i x$ for $i \in \{0, \dots, m-1\}$ and $J_k x = 0$ for $k \in \{0, \dots, n-1\} - \{0, \dots, m-1\}$ is a B_n -algebra.

LEMMA 1.3. *If the filter F is maximal, then \mathcal{A}/F is isomorphic to \mathbb{B}_m^f for suitable f and m , where $2 \leq m \leq n$ if n is odd, and $m = 2k, 2 \leq m \leq n$ if n is even.*

Representation theorem. Every B_n -algebra \mathcal{A} is isomorphic to the subdirect product of algebras \mathbb{B}_m^f , where $2 \leq m \leq n$ if n is odd, and $m = 2k, 2 \leq m \leq n$ if n is even.

The formulae of the logic B_n are constructed by means of propositional variables and the connectives $\cup, \cap, \sim, J_0, \dots, J_{n-1}$ (where \cup, \cap are binary and $\sim, J_0, \dots, J_{n-1}$ are unary) in the usual way. We shall denote them by $\alpha, \beta, \gamma, \dots$. Formulae of the form $J_i \alpha, \sim J_i \alpha$ will be denoted by ξ, η, ζ, \dots . We introduce the following abbreviations:

$$\begin{aligned} \alpha \supset \beta &\equiv \sim \alpha \cup \beta, 0 \equiv \sim J_{n-1} \alpha \cap J_{n-1} \alpha, \\ 1 &\equiv \sim J_{n-1} \alpha \cup J_{n-1} \alpha, \\ \alpha &\equiv \beta \equiv \bigcap_{i=0}^{n-1} ((J_i \alpha \supset J_i \beta) \cap (J_i \beta \supset J_i \alpha)). \end{aligned}$$

Now we shall construct the calculi B_n by giving a finite number of axiom schemes and inference rules of modus ponens:

- $B_n 1. (\alpha \cup \alpha) \equiv \alpha$
 $B_n 2. (\alpha \cup \beta) \equiv (\beta \cup \alpha)$
 $B_n 3. (\alpha \cup (\beta \cup \gamma)) \equiv ((\alpha \cup \beta) \cup \gamma)$
 $B_n 4. (\alpha \cap (\beta \cup \alpha)) \equiv ((\alpha \cap \beta) \cup (\alpha \cap \gamma))$
 $B_n 5. \sim(\sim(\alpha)) \equiv \alpha$
 $B_n 6. \sim(\sim J_{n-1}\alpha \cup J_{n-1}\alpha) \equiv (\sim J_{n-1}\alpha \cap J_{n-1}\alpha)$
 $B_n 7. \sim(\alpha \cup \beta) \equiv (\sim \alpha \cap \sim \beta)$
 $B_n 8. ((\sim J_{n-1}\alpha \cap J_{n-1}\alpha) \cup \beta) \equiv \beta$
 $B_n 9. J_{n-1}\xi \equiv \xi$
 $B_n 10. J_0\xi \equiv \sim \xi$
 $B_n 11. J_i\xi \equiv (\sim J_{n-1}\alpha \cap J_{n-1}\alpha), \quad 0 < i < n-1$
 $B_n 12. J_i(\sim \alpha) \equiv J_{n-1-i}\alpha, \quad 0 \leq i \leq n-1$
 $B_n 13. J_i\alpha \equiv (J_0\alpha \cup \dots \cup J_{i-1}\alpha \cup J_{i+1}\alpha \cup \dots \cup J_{n-1}\alpha) \quad 0 \leq i \leq n-1$
 $B_n 14. J_i\alpha \cup \sim J_i\alpha \equiv (J_{n-1}\alpha \cup \sim J_{n-1}\alpha), \quad 0 \leq i < n-1$
 $B_n 15. ((J_i\alpha \cup J_k\beta) \cap J_i\alpha) \equiv J_i\alpha, \quad 0 \leq i, k \leq n-1$
 $B_n 16. (\alpha \cup J_i\alpha) \equiv \alpha, \quad n-1 \geq i \geq n-1-i.$
 $B_n 17. J_k(\alpha \cup \beta) = \bigcup_{j=0}^k (J_k\alpha \cap J_j\beta) \cup \bigcup_{i=0}^k (J_i\alpha \cap J_k\beta), \quad 0 \leq k < [\frac{n}{2}]$
 $B_n 18. J_k(\alpha \cup \beta) = \bigcup_{i=k}^{n-1} (J_i\alpha \cap J_k\beta) \cup \bigcup_{i=k}^{n-1} (J_k\alpha \cap J_i\beta) \cup$
 $\bigcup_{i=k}^{n-1} (J_i\alpha \cap J_k \sim \beta) \cup \bigcup_{i=k}^{n-1} (J_k\alpha \cap J_i \sim \beta) \cup$
 $\bigcup_{i=k}^{n-1} (J_i \sim \alpha \cap J_k\beta) \cup \bigcup_{i=k}^{n-1} (J_k \sim \alpha \cap J_i\beta),$
 $n-1 \geq k \geq [\frac{n+1}{2}].$
 $B_n 19. \xi \supset (\eta \supset \zeta)$
 $B_n 20. (\xi \supset (\eta \supset \zeta)) \supset ((\xi \supset \eta) \supset (\xi \supset \zeta))$
 $B_n 21. (\xi \cap \eta) \supset \xi$
 $B_n 22. (\xi \cap \eta) \supset \eta$
 $B_n 23. (\xi \supset \eta) \supset ((\xi \supset \zeta) \supset (\xi \supset (\eta \cap \zeta)))$
 $B_n 24. \xi \supset (\xi \cup \eta)$
 $B_n 25. \eta \supset (\xi \cup \eta)$
 $B_n 26. (\xi \supset \zeta) \supset ((\eta \supset \zeta) \supset ((\xi \cup \eta) \supset \zeta))$
 $B_n 27. (\xi \supset \eta) \supset (\sim \eta \supset \sim \xi)$
 $B_n 28. \xi \supset \sim \sim \xi$
 $B_n 29. \sim \sim \xi \supset \xi$

Inference rule: $\frac{\alpha, \alpha \supset \beta}{\beta}$

A formula α is said to be a tautology if α considered as an algebraic polinom has the value 1 for each assignment of variables by elements of the algebra \mathbb{B}_n .

COMPLETENESS THEOREM. *For each formula α , α is a theorem $B_n(\vdash B)$ iff α is a tautology.*

Note that the Bochvar's 3-valued logic has already been formalized in several different ways [2,3,5]. Some authors treat this logic as the nonsense-logic or the logic of significance.

References

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