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A SIMPLER PROOF OF SAHLQVIST'S THEOREM ON COMPLETENESS OF MODAL LOGICS

This is a preliminary report of a part of the paper [6], not submitted for publication yet.

We assume familiarity with standard notions and notation in propositional modal logic (cf. e.g. [4] or [5]). As usual, we say that a (propositional normal modal) logic L is *complete* if $\vdash_L P$ iff $\models_L P$ (where $\models +LP$ means that the formula P is valid in all frames for L). Also, we say that a formula P *corresponds* to a first-order formula χ_P if for each frame F, $F \models P$ iff P satisfies the elementary property expressed by χ_P. In [5], Sahlqvist proves the following:

Theorem. *Let S be any modal formula which is equivalent to a conjunction of formulas of the form $\Box^m(Q \to P)$ where*

1. *P is positive,*
2. *after eliminating $\to$ from Q and rewriting Q with $\neg$ occurring only in front of propositional variables, no positive occurrence of a variable is in a subformula of Q of the form $P_1 \vee P_2$ or $\Diamond P_1$ within the scope of some $\Box$.*

Then KS (= the logic obtained from K adding S as an axiom schema) is complete and S corresponds to a first-order formula χ_S effectively obtainable from S.

We believe that the range of applications of this theorem is so wide (e.g. each of $D, T, B, S4, K4, S5, \ldots$ is included) that the effort for simplifying its proof is worthwhile. Our simplification is based on the introduction of a topology in each first-order frame, as done in [6] to which we refer for more details. Any first-order frame $F = (X, R, T)$ (T is a boolean subalgebra of 2^X which is closed under an operation R^* defined by $R^*C = \{x \in X : xRy \Rightarrow y \in C\}$, for each $C \subseteq X$) is topologized taking T as a base for open subsets (and hence also for closed subsets, since each element of T becomes open and closed). Then we say that (X, R, T) is *descriptive* if the topology induced by T is compact and Hausdorff and if $R^n x = \{y : xR^n y\}$ is closed for each n and each $x \in X$ (the equivalence of this definition to that in [2] is almost obvious). The usefulness of this notion rests on two facts:

a) each first-order frame is equivalence to a descriptive frame;
b) all first-order canonical frames are descriptive.

The main method to prove completeness of a logic L is to show that it is canonical, i.e. that L holds on its canonical frame (as a matter of fact, by this method one obtains strong completeness, i.e. completeness for the deductive system $\vdash_L$). Combining this with b) and recalling that each logic holds on its first-order canonical frame, to show completeness of KS it is enough to show that S is *persistent*, i.e. that for each descriptive $F = (X, R, T)$, $(X, R, T) \models S$ implies $(X, R) \models S$.

Now let us say that a formula is *plain* if it is obtained starting from formulas of the form $\Box^m p_i$ and negative formulas by applying only $\wedge$ and $\Diamond$. Then, with no difficulty but with some patience, one can prove that any formula Q satisfying the condition 2. of the theorem is equivalent to a disjunction of plain formulas. So the formula S of the theorem may be rewritten as a conjunction of formulas of the form $\Box^m(Q \to P)$, where Q is plain and P is positive. Therefore the proof of the theorem reduces to a proof of:

LEMMA 1. *If Q is a plain formula, P a positive formula and $m \geq 0$, then $S = \Box^m(Q \to P)$ is persistent and corresponds to a first-order formula χ_S.*

The proof for correspondence is quite similar to that in [5], but we have to repeat it here since persistence is obtained on the way. Spelling out definitions, one easily finds out that with each modal formula $P(p_1, \ldots, p_k)$ one can associate a formula $c_p(u, S_1, \ldots, S_k)$, which is first-order except for unary predicate variables S_i and in which u is the only free individual variable, such that for any frame $F = (X, R, T)$ and any $x \in X$, $F \models P$ iff $(\forall S_1 \ldots S_k \in T) c_p(u, S_1, \ldots, S_k)$ holds in (X, R) at x (cf. [3], pp. 34–35).

Using the abbreviations: $x \in S_i$ for $S_i(x)$, $R^n x \subseteq S_i$ for $xR^n y \to y \in S_i$ and $\bigcup_{j \leq k} R^n x_j \subseteq S_i$ for $R^n x_1 \subseteq S_i \wedge \ldots \wedge R^n x_k \subseteq S_i$, one can show by induction that when Q is plain, $c_Q(u)$ may be written in the form:

$$(*)\ \exists z_1 \ldots \exists z_n [A \wedge \bigcup_{j \leq m_1} R^{n_{j,1}} y_j \subseteq S_1 \wedge \ldots \wedge \bigcup_{j \leq m_k} R^{n_{j,k}} y_j \subseteq S_k \wedge$$
$$c_{N_1}(y_{h_1}, S_1, \ldots, S_k) \wedge \ldots \wedge c_{N_l}(y_{h_l}, S_1, \ldots, S_k)]$$

where A is a conjunction of atomic formulas of the form $y_i R y_j$: all y's are among $z_1, \ldots, z_n$ and u; $N_1, \ldots, N_l$ are negative formulas and all indices may be equal to zero.

Hint. If $Q = \Box^m p_i$, then c_Q is $R^m u \subseteq S_i$, and if Q is negative the claim is trivial. Inductive cases (only $\wedge$ and $\Diamond$!) are mere manipulations. To get a grasp of what is going on, we suggest the reader to try on an easy formula.

Now let $S = \Box^m(Q \to P)$ where Q is plain and P is positive. Then $c_S(u)$ is $\forall w(uR^m w \to (c_Q(w) \to c_p(w)))$. Using $(*)$ where we write U_i for $\bigcup_{j \leq m_i} R^{n_{j,i}} y_j$) we can write c_S as

$$\forall w \forall z_1 \ldots z_n (uR^m w \wedge A \to (\bigwedge_{j \leq k} (U_i \subseteq S_i) \to (\bigvee_{i \leq l} c_{N'_i}(y_{h_i}) \vee c_p(w))))$$

where each N'_i is positive. For convenience, let $A' = uR^m w \wedge A$ and $B(S_1, \ldots, S_k) = \bigvee_{i \leq l} c_{N'_i}(y_{h_i}) \vee c_p(w)$. So for any first-order frame $F = (X, R, T)$, we have that: $F \models_x S$ iff

$$(**)\ (\forall S_1 \ldots S_k \in T) \forall w \forall z_1 \ldots z_n (A' \to (\bigwedge_{i \leq k} (U_i \subseteq S_i) \to B(S_1, \ldots, S_k)))$$

holds at x.

To prove correspondence, suppose $(X, R) \models_x S$. Then evaluating $S_1, \ldots, S_k$ in $(**)$ on $U_1, \ldots, U_k$, we have that

$$\chi_S : \forall w \forall z_1 \ldots z_n (A' \to B(U_1, \ldots, U_k))$$

holds at x. Note that χ_S is a first-order formula.

Conversely, suppose χ_S holds at x and suppose A' holds for some given values of $w, z_1, \ldots, z_n$. Then for such values $B(U_1, \ldots, U_k)$ holds and hence, since B comes from a positive modal formula, also $B(S_1, \ldots, S_k)$ holds whenever $U_i \subseteq S_i$. So we have proved that

$(**)'$ $(\forall S_1 \ldots S_k \subseteq X)\forall w \forall z_1 \ldots z_n(A' \to (\bigwedge_{i \leq k}(U_i \subseteq S_i) \to B(S_1, \ldots, S_k)))$

holds at x. But $(XX)'$ is nothing else than $(**)$ written for the frame (X, R), and therefore $(X, R) \models_x S$.

To prove persistence, it is enough to show that on any descriptive $F = (X, R, T)$, $(X**)$ implies χ_S. For in this case also $(**)'$ holds by the above correspondence of S to χ_S and hence also $(X, R) \models_x S$.

The following fact is the key of our simplification (cf. also [2], section 14):

Lemma 2. *Let P be a positive formula, (X, R, T) a descriptive frame and Y any subset of X. Then*

$$\bigcap_{Y \subseteq C \in T} \{x \in X : c_p(., C, .) \text{ holds at } x\} = \{x \in X : c_p(., \overline{Y}, .) \text{ holds at } x\},$$

where $\overline{Y}$ is the (topological) closure of Y.

Proof. By induction on the structure of P (first write P without $\neg$). The step for $\Diamond$ is substantially the following fact (cf. [1], Lemma 3; compactness of the frame is essential here): for any filter H of closed subsets of $X, R^{-1}(\bigcap H) = \bigcap_{C \in H} R^{-1}C$.

Now rewrite $(**)$ as

$$\forall w \forall z_1 \ldots z_n(A' \to (\forall S_1 \ldots S_k \in T)(\forall_{i \leq k}(U_i \subseteq S_i) \to B(S_1, \ldots, S_k))).$$

Then by Lemma 2. the subformula to the right of A' holds iff $B(\overline{U}_1, \ldots, \overline{U}_k)$ holds. By the definition of descriptive frames, each U_i is a closed subset of X, hence $U_i = \overline{U}_i$ and χ_S holds.

As a final comment, note that we have proved something more than required: correspondence and persistence of S are in fact proved to hold at each point (following [7], we would say that they hold locally).

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