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MORE ABOUT REFERENTIAL MATRICES

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The present note being complementary to [1], I shall only briefly recall the key notions to be exploited here, and for more details the reader is advised to consult [1].

By a propositional logic we mean a couple $(\underline{L}, C)$, where $\underline{L}$ is a *propositional language* (algebra of formulas) and C a *structural consequence* defined on $\underline{L}$. A couple $W = (\underline{A}, D)$ is said to be a *referential matrix* for the language $\underline{L}$ iff there exists a non-empty set T such that the following two conditions are satisfied:

i. $\underline{A}$ is an abstract algebra similar to $\underline{L}$, whose all elements belong to $\{0,1\}^T$, i.e. they are mappings from T into the two-element set $\{0,1\}$.

ii. $D = \{\{a \in \underline{A} : a(t) = 1\} : t \in T\}$.

Let $(\underline{L}, C)$ be a propositional logic. The main theorem of [1] says that a referential matrix W such that $C = Cn_W$ (Cn_W being the consequence operation determined by W) exists iff C is self-extensional, i.e.

(a) $C(\alpha) = C(\beta)$

implies

(a′) $C(\varphi(\alpha/p)) = C(\varphi(\beta/p))$, for all formulas φ of $\underline{L}$ and all propositional variables p.

THEOREM. *Let* $(\underline{L}, C)$ *be a propositional logic. A referential matrix* W *such that* $C(\emptyset) = Cn_W(\emptyset)$ *exists iff* C *in weakly self-extensional, i.e.*

(b) $\alpha, \beta \in C(\emptyset)$

implies

(b′) $\varphi(\alpha/p) \in C(\emptyset)$ *iff* $\varphi(\beta/p) \in C(\emptyset)$.

PROOF (AN OUTLINE). (→). Suppose that $C(\emptyset) = Cn_W(\emptyset)$. Assume (b). This implies that $Cn_W(\alpha) = Cn_W(\beta)$. But Cn_W is self-extensional. Hence $\varphi(\alpha/p) \in Cn_W(\emptyset) = C(\emptyset)$ iff $\varphi(\beta/p) \in Cn_W(\emptyset) = C(\emptyset)$, which yields (b′).

(←) Define a consequence operation C_0 on $\underline{L}$ by the following condition. For each $\alpha \in \underline{L}$, each $X \subseteq \underline{L}$, $\alpha \in C(X)$ iff α is provable from $X \cup C(\emptyset)$ by means of the following rule:

(R) If $\alpha, \beta \in C(\emptyset)$, then from $\varphi(\alpha/p)$ (any formula of this form) infer $\varphi(\beta/p)$.

Observe that the rule R is structural. Indeed, given any couple of formulas $\varphi(\alpha/p)$, $\varphi(\beta/p)$ and any endomorphism $e \in Hom(\underline{L}, \underline{L})$ one may find a formula ψ such that $e\varphi(\alpha/p) = \psi(e\alpha/p)$ and $e\varphi(\beta/p) = \psi(e\beta/p)$. Of course if $\alpha, \beta \in C(0)$, then also $e\alpha, e\beta \in C(\emptyset)$. hence if $\varphi(\beta/p)$ is derivable by R from $\varphi(\alpha/p)$ so is $e\varphi(\beta/p)$ from $e\varphi(\alpha/p)$.

The structurality of R together with the fact that $C(\emptyset)$ is closed under substitutions implies that C_0 is structural. Moreover, since, as one may easily see, $C(\emptyset)$ is closed under R, we have $C(\emptyset) = C_0(\emptyset)$.

The next step in the argument consists in showing that C_0 is self-extensional. In order to see this, it is enough to notice that whenever $C_0(\alpha) = C_0(\beta)$ and $\alpha_1, \ldots, \alpha_n$ is a proof of α from $\{\beta\} \cup C(\emptyset)$ by means of R, then for each formula of the form $\varphi(\alpha/p)$, the proof $\alpha_1, \ldots, \alpha_n$ can be transformed to the proof $\alpha'_1, \ldots, \alpha'_n$ of $\varphi(\alpha/p)$ from $\{\varphi(\beta/p)\} \cup C(\emptyset)$ by means of R in the following way. If $\alpha_i \in C(\emptyset)$, put $\alpha'_i = \alpha_i$. If $\alpha_i = \beta$, put $\alpha'_i = \varphi(\beta/p)$. Finally, if α_i results from α_j, $j < i$, by R, define α'_i to be the formula which results from α'_j by the application of R which involves the formulas from $C(\emptyset)$ used in order to get α_i from α_j and which is ordered out with respect to the same variable as in the case of α_i and α_j.

The self-extensionality of C_0 implies that there exists a referential matrix W such that $C_0 = Cn_W$. But $C_0(\emptyset) = C(\emptyset)$ and hence we obtain $C(\emptyset) = Cn_W(\emptyset)$, concluding the proof.

References

[1] Ryszard Wójcicki, *Referential matrix semantics for propositional calculi*, **Bulletin of the Section of Logic** 8 (1979), pp. 170–176.

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