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ON FINITELY BASED CONSEQUENCE DETERMINED BY A DISTRIBUTIVE LATTICE

Let $\underline{F} = \langle F, \wedge \vee \rangle$ be a free algebra in the class of all algebras of type $\langle 2, 2 \rangle$ freely generated by the set of variables $V = \{p_0, p_1, p_2, \ldots\} = \{p_i : i \in N\}$. The elements of the set F will be called formulas. We shall use the Latin lower case letters x, y, z for formulas and U, X, Y for sets of formulas. The symbol $V(U)$ denotes the set of all variables occurring in formulas of U. By R we denote the set of the following rules:

$R1.\ \dfrac{x \wedge y}{x}$, $\qquad R7.\ \dfrac{x \vee (y \vee z)}{(x \vee y) \vee z}$,

$R2.\ \dfrac{x \wedge y}{y \wedge x}$, $\qquad R8.\ \dfrac{(x \vee y) \vee z}{x \vee (y \vee z)}$,

$R3.\ \dfrac{x, y}{x \wedge y}$, $\qquad R9.\ \dfrac{x \vee (y \wedge z)}{(x \vee y) \wedge (x \vee z)}$,

$R4.\ \dfrac{x}{x \vee y}$, $\qquad R10.\ \dfrac{(x \vee y) \wedge (x \vee z)}{x \vee (y \wedge z)}$,

$R5.\ \dfrac{x \vee y}{y \vee x}$, $\qquad R11.\ \dfrac{x \wedge (y \vee z)}{(x \wedge y) \vee (x \wedge z)}$,

$R6.\ \dfrac{x \vee (x \vee y)}{x \vee y}$, $\qquad R12.\ \dfrac{x \vee x}{x}$.

The set of rules R determines a consequence operation $Cn_R : 2^F \to 2^F$ such that for every $X \subseteq F$, $Cn_R(X)$ is the smallest subset of F containing X and closed under the rules of R.

Define the relation $\vdash_R \subseteq F \times F$ as follows:

$$x \vdash_R y \Leftrightarrow y \in Cn_R(\{x\}), \text{ for every } x, y \in F.$$

Then we obtain the following

LEMMA 0. *For every $x, y, z \in F$:*

(i) $(x \wedge y) \vee x \vdash_R x$,
(ii) $(x \wedge y) \vee z \vdash_R y \vee z$,
(iii) $(x \wedge y) \vee (x \wedge z) \vdash_R x \wedge (y \vee z)$.

PROOF. (i)

(1)	$(x \wedge y) \vee x \vdash_R x \vee (x \wedge y)$	$\{R5\}$
(2)	$x \vee (x \wedge y) \vdash_R (x \vee x) \wedge (x \vee y)$	$\{R9\}$
(3)	$(x \vee x) \wedge (x \vee y) \vdash_R x \vee z$	$\{R1\}$
(4)	$x \vee x \vdash_R x$	$\{R12\}$.

As analogous to (i), the proofs of (ii) and (iii) are left out.

Let $\underline{K} = \langle K, \cap, \cup \rangle$ be a distributive lattice with 0 and 1 such that $0 \neq 1$, and let $\mathbb{K} = \langle \underline{K}, \{1\} \rangle$. The class of all such matrices $\mathbb{K}$ will be denoted by $\mathcal{V}$. Every matrix $\mathbb{K} \in \mathcal{V}$ determines a consequence operation $C_{\mathbb{K}}$ as follows:

$$x \in C_{\mathbb{K}}(X) \Leftrightarrow \forall_{hHom(\underline{F},\underline{K})}[h(X) \subseteq \{1\} \Rightarrow h(x) = 1],$$

for every $x \in F$ and $X \subseteq F$, (cf. [1]),

In this paper we show that $Cn_R = C_{\mathbb{K}}$, for every $\mathbb{K} \in \mathcal{V}$.

The following lemma holds.

LEMMA 1. $Cn_R \leq C_{\mathbb{K}}$*, for every* $\mathbb{K} \in \mathcal{V}$*.*

We omit an easy proof of this lemma.

Let AE be a smallest set satisfying the following conditions:

a. $V \subseteq AE$
b. $x, y \in AE \Rightarrow x \vee y \in AE$.

Now, we define the function $d : F \to 2^F$ as follows:

a. $d(x) = \{x\}$, if $x \in V$,
b. $d(x \wedge y) = \{x \wedge y\}$,
c. $d(x \vee y) = d(x) \cup d(y)$,

for every $x, y \in F$.

Then we obtain

LEMMA 2. *For any* $x \in F - AE$ *there exist* $y, z \in F$ *such that* $y \wedge z \in d(x)$.

The proof of this lemma is straightforward.

Now let s be the function $s : F \to N$ defined as follows:

a. $s(x) = 0$, if $x \in V$,

b. $s(x \wedge y) = s(x \vee y) = s(x) + s(y) + 1$,

for every $x, y \in F$.

Then the following holds.

LEMMA 3. *If* $d(x) = \{z_1, z_2, \ldots, z_n\}$ *and* $n \geq 2$, *then for every permutation* $i_1, i_2, \ldots, i_n$ *of the sequence* $1, 2, \ldots, n$ *we have:*

(i) $Cn_R(\{x\}) = Cn_R(\{z_{i_1} \vee (z_{i_2} \vee \ldots \vee (z_{i_{n-1}} \vee z_{i_n}) \ldots)\})$,

(ii) $V(\{x\}) = V(\{z_{i_1} \vee (z_{i_2} \vee \ldots \vee (z_{i_{n-1}} \vee z_{i_n}) \ldots)\})$,

(iii) $s(z_{i_1} \vee (z_{i_2} \vee \ldots \vee (z_{i_{n-1}} \vee z_{i_n} \ldots)) \leq s(x)$.

PROOF. (i) The lemma 3(i) is obtained by applying the rules $R4, R5, R7, R8$ and $R5 - R8$.

Easy proofs of (ii) and (iii) can be omitted.

LEMMA 4. *For every* $x \in F$ *there exists a finite set* $U \subseteq AE$ *such that:*

(i) $Cn_R(\{x\}) = Cn_R(U)$,

(ii) $V(\{x\}) = V(U)$.

PROOF. If $x \in V$, then the lemma is obvious. Assume inductively that for every formula w such that $s(w) < s(x)$ the considered lemma holds. Let $x = y_1 \vee z_1$. If $x \in AE$, then the lemma holds, for $U = \{x\}$. Let $x \in F - AE$. Hence, by Lemma 2 and Lemma 3, for some $y, z, x_1 \in F$ we have:

(1) $y \wedge z \in d(x)$,

(2) $Cn_R(\{x\}) = Cn_R(\{y \wedge z) \vee x_1\})$,

(3) $V(\{x\}) = V(\{y \wedge z) \vee x_1\})$,

(4) $s((y \wedge z) \vee x_1) \leq s(x)$.

From (2) and (4), applying the rules: $R9, R10, R11, R1, R2, R3, R5$, and Lemma 0(iii), we obtain:

(5) $Cn_R(\{x\}) = Cn_R(\{y \vee x_1, z \vee x_1\})$,
(6) $s(y \vee x_1) < s(x)$ and $s(z \vee x_1) < s(x)$.

From (6)and from the inductive hypothesis, applying (5) we have:

(7) $Cn_R(\{x\}) = Cn_R(U_1 \cup U_2)$,
(8) $V(\{x\}) = V(U_1 \cup U_2)$,

for some finite subsets U_1, U_2 of AE.

Thus the proof for $x = y_1 \vee z_1$ is completed. For $x = y \wedge z$ it is quite simple.

LEMMA 5. *If $x \in AE$, $U \subseteq AE$ and $x \notin Cn_R(U)$, then there exists a set $Y \subseteq V$ such that:*

(i) $x \notin Cn_R(Y)$,
(ii) $U \subseteq Cn_R(Y)$.

PROOF. Let $x \in AE$, $U \subseteq AE$ and

(1) $x \notin Cn_R(U)$.

Put

(2) $V_z =_{df} V(\{z\}) - V(\{x\})$, for every $z \in U$.

Then, by (1), we have

(3) $V_z \neq \emptyset$, for every $z \in U$.

Hence, for $Y_{df}\{p_{i_z} : z \in U\}$, where i_z is some natural number such that $p_{i_z} \in V_z$, we obtain:

(4) $x \notin Cn_R(Y)$ and $U \subseteq Cn_R(Y)$.

This completes the proof of Lemma 5.

Now, let $K_2 = \langle \{0, 1\}, \cap, \cup \rangle$ and $\mathbb{K}_2 = \langle K_2, \{1\} \rangle, 0 \neq 1$. We shall prove the following

THEOREM. $C_{\mathbb{K}_2} \leq Cn_R$.

PROOF. Let $X \subseteq F$ and

(1) $z \notin Cn_R(X)$.

According to Lemma 4, for some finite set $U \subseteq AE$ we have:

(2) $Cn_R(\{z\}) = Cn_R(U)$,

(3) $V(\{z\}) = V(U)$.

Hence for some $x_1 \in U$

(4) $x_1 \notin Cn_R(X)$.

By Lemma 4 we also have

(5) $Cn_R(X) = Cn_R(U_1)$, for some $U_1 \subseteq AE$.

From (4), (5) and Lemma 5

(6) $x_1 \notin Cn_R(Y_1)$ and
(7) $X \subseteq Cn_R(Y_1)$,

for some $Y_1 \subseteq V$.
Now, we define the function $v : V \to \{0,1\}$ as follows:

(8) $v(p_i) = \begin{cases} 1, & \text{if} \quad p_i \in Y_1 \\ 0, & \text{if} \quad p_i \in V - Y_1. \end{cases}$

By Lemma 1 and by the fact that $R3, R4, R5 \in R$ we have

(9) $y \in Cn_R(Y_1) \Leftrightarrow h^v(y) = 1$, for every $y \in F$,

where h^v is the extension of the function v to the homomorphism of the algebra $\underline{F}$ to K_2.
According to (2), (4) and (6) we obtain

(10) $z \notin Cn_R(Y_1)$.

Hence, by (9) and (7): $h^v(X) \subseteq \{1\}$ and $h^v(z) = 0$, which completes the proof of the theorem.

From Lemma 1 and from Theorem we have

Corollary. *$Cn_R = C_{\mathbb{K}}$, for every $\mathbb{K} \in \mathcal{V}$.*

References

[1] J. Łoś and R. Suszko, *Remarks on sentential logics*, **Indagationes Mathematicae**, Vol. 20 (1958), pp. 177–183.

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