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## A THEOREM OF THE DEGREE OF COMPLEXITY OF SOME SENTENTIAL LOGICS

§1. This paper is a contribution to matrix semantics for sentential logics as presented in Łoś and Suszko [1] and Wójcicki [3], [4].

A generalization of Lindenbaum completeness lemma says that for each sentential logic  $(\underline{L}, C)$  there is a class  $\mathbb{K}$  of matrices of the form  $(\underline{A}, D)$  such that the class is (strongly) adequate for the logic, i.e.,  $C = Cn_{\mathbb{K}}$ .

In connection with this general completeness lemma the following questions arise:

- (1) Under what conditions on a logic  $(\underline{L}, C)$  does there exist a class  $\mathbb{K}$  of matrices such that:
  - (i)  $C = Cn_{\mathbb{K}}$  and (ii) there is no class  $\mathbb{K}'$  of lesser cardinality than  $\mathbb{K}$  such that  $C = Cn_{\mathbb{K}'}$ .
- (2) Given a logic  $(\underline{L}, C)$ , find the least cardinal number  $m$  such that  $C = Cn_{\mathbb{K}}$  for some class  $\mathbb{K}$  of cardinality  $m$ . Such an  $m$  always exists and is called a *degree of complexity* of  $(\underline{L}, C)$  (cf. [4], p 51).

We say that  $X$  and  $Y$  ( $X, Y \subseteq L$ ) are *mutually uniform* in a logic  $(\underline{L}, C)$ , in symbols  $X \simeq_C Y$ , iff

$$A \in C(Z \cup X') \Leftrightarrow A \in C(Z \cup Y')$$

for all  $X'$  obtained from  $X$  by substitution of distinct variables for variables, all  $Y'$  obtained similarly from  $Y$ , and all  $Z \cup \{A\} \subseteq L$  such that  $Z \cup \{A\}$  is separated from  $X' \cup Y'$  (cf. [3], [4], p. 61).

One can prove that  $\simeq_C$  is an equivalence relation.

Let  $Th(C)$  be the set of all consistent systems (theories) of  $(\underline{L}, C)$ . A logic  $(\underline{L}, C)$  is said to be *m-uniform* iff  $m$  is the cardinality of the quotient set  $Th(C)/\simeq_C$  (cf. [4], p. 64).

A logic is *regular* iff the members of any separated family  $\{X_i\}$  of mutually uniform sets are mutually uniform with  $\bigcup X_i$ .

It is known that if the degree of complexity of a logic is 1, then the logic is 1-uniform and regular.

WÓJCICKI'S THEOREM (cf. [3], [4], p. 64). *If a consistent logic is m-uniform and regular, then its degree of complexity is not greater than m.*

§2. It is known that the degrees of complexity of Johansson's minimal logic and of modal logics associated with modal systems S1-S3 are greater than 1. In this note we give necessary conditions for an extension of Johansson's minimal logic to have the degree of complexity equal to 2.

DEFINITION 1. Assume that  $(\underline{L}, C_1)$  and  $(\underline{L}, C_2)$  are logics in the same language with negation  $\sim$  and  $C_2 \leq C_1$ . We say that  $(\underline{L}, C_2)$  has the *Glivenko property with respect to*  $(\underline{L}, C_1)$  iff  $A \in C_1(X)$  implies  $\sim\sim A \in C_2(X)$ , all  $X \cup \{A\} \subseteq L$ .

We assume that up to the end of this abstract the language  $\underline{L}$  has the connectives  $\rightarrow, \wedge, \vee$ , and a constant symbol  $\perp$ . In this language by  $C_J, C_I, C_K$  we shall denote, respectively, Johansson's minimal logic, intuitionistic and classical logics. By  $C_J^*$  we denote the extension of  $C_J$  by the following axiom scheme:

$$(*) \quad ((\perp \rightarrow A) \rightarrow \perp) \rightarrow \perp.$$

Let us put:  $\sim A =_{df} A \rightarrow \perp$ .

LEMMA 1. *Each formula of the form  $\sim\sim (A \Rightarrow B) \rightarrow (A \rightarrow \sim\sim B)$  is a theorem of  $C_J$ .*

LEMMA 2. *If  $C \geq C_J^*$ , then  $C$  has the Glivenko property with respect to  $C_K$ .*

From Lemma 2 it follows that  $C_J^*$  is the least logic being a strengthening of  $C_J$  and having the Glivenko property with respect to  $C_K$  (cf. [2], p. 46).

LEMMA 3. *Let  $e$  be a substitution transforming variables of  $\underline{L}$  into  $\{p \rightarrow p, \perp\}$ . Then for all  $X \subseteq L$  we have:*

$$C_J(eX) = C_J(\emptyset) \text{ or } C_J(eX) = C_J(\perp).$$

THEOREM. *The degree of complexity of every logic  $(\underline{L}, C)$ , such that  $C(\perp) \neq L$ ,  $\perp \notin C(\emptyset)$  and  $C \geq C_J^*$  equals 2.*

PROOF.. By Wójcicki's theorem and by inspection of  $\simeq_C$  with the help of the above lemmas.

### §3. Open Problems.

1. Find simple matrices  $M_1, M_2$  such that  $C_J^* = Cn_{\{M_1, M_2\}}$ .
2. What is the degree of complexity of the minimal logic of Johansson?
3. What are the degrees of complexity of the modal logics associated with the modal systems S1-S3 of Lewis?

## References

- [1] J. Łoś, R. Suszko, *Remarks on sentential logics*, **Indagationes Mathematicae** 20 (1958), pp. 177-183.
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- [3] R. Wójcicki, *Some remarks of the consequence operation in sentential logics*, **Fundamenta Mathematicae** 68 (1970), pp. 269-279.
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