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MODAL LOGICS BETWEEN $S4.2$ and $S4.3$

I

The logics $S4.2$ and $S4.3$ are formed by adding to $S4$ the axioms

$G.\quad MLp \supset LMp$

and $Lem.\quad L(Lp \supset q) \vee L(Lq \supset p)$

respectively. As is well known, $S4.3$ properly contains $S4.2$. It is also a standard result that $S4.2$ is characterized by the class of all frames (W, R) in which R is reflexive, transitive and *convergent* in the sense that

$$(\forall x, y, z \in W)((xRy \wedge zRz) \supset (\exists w \in W)(yRw \wedge zRw))$$

and that $S4.3$ is characterized by the class of all frames in which R is reflexive, transitive and *connected* in the sense that

$$(\forall x, y, z \in W)((xRy \wedge xRz) \supset (yRz \vee zRy)).$$

This paper defines an infinite sequence of logics properly between $S4.2$ and $S4.3$ and shows what classes of frames characterize them.

II

For each $n \geq 0$, let Lem_n be

$$L(Lp_0 \supset a_n) \vee L(Lp_1 \supset p_0)$$

where a_n is defined inductively as follows:

a_0 is p_1
a_{k+1} is $p_{k+1} \supset L(p_{k+1} \vee a_k)$.

Then, again for each $n \geq 0$, we define $S4.3_n$ as $S4.2 + Lem_n$.

In particular instances we shall write p for p_0, q for p_1, etc., and replace $q \vee q$ by q. Thus Lem_0 will be $L(Lp \supset q) \vee L(Lq \supset p)$. Lem_1 will be $L(Lp \supset (q \supset Lq)) \vee L(Lq \supset p)$. Lem_2 will be $L(Lp \supset (r \supset L(r \vee (q \supset Lq)))) \vee L(Lq \supset p)$, and so forth. Clearly Lem_0 is the original Lem, and hence $S4.3_0$ is simply $S4.3$.

I shall consider $S4.3_1$ in some detail and then show in outline how to generalize the results for the whole sequence of $S4.3_n$ logics.

III

THEOREM 1. *$S4.3$ contains $S4.3_1$.*

PROOF. Clearly it is sufficient to show that $\vdash_{S4.3} Lem_1$. We do so as follows ($\underline{L}$ is the rule: $\vdash \alpha \supset \beta \rightarrow \vdash L\alpha \supset L\beta$):

(1)	$L(Lp \supset (q \supset Lq)) \vee L(L(q \supset Lq) \supset p)$	$[Lem(q \supset Lq/q)]$
(2)	$Lq \supset L(q \supset Lq)$	$[S4]$
(3)	$L(L(q \supset Lq) \supset p) \supset L(Lq \supset p)$	$[(2), PC, \underline{L}]$
(4)	$L(Lp \supset (q \supset Lq)) \vee L(Lq \supset p)$ (Lem_1)	$[(1), (3), PC]$

THEOREM 2. *$S4.2$ does not contain $S4.3_1$.*

PROOF. The frame of Figure 1, with R assumed to be reflexive and transitive, is clearly convergent, and hence a frame for $S4.2$. But Lem_1 is false at x in the model on this frame in which $V(p) = \{y, w, v\}$ and $V(q) = \{y, z, v\}$.

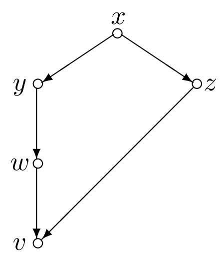

Fig. 1

THEOREM 3. *$S4.3_1$ does not contain $S4.3$.*

PROOF. The frame of Figure 2 (R reflexive and transitive) is a frame for $S4.3_1$, but *Lem* is false at x in the model on this frame in which $(V(p) = \{y, w\}$ and $V(q) = \{z, w\}$.

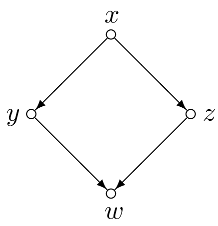

Fig. 2

THEOREM 4. *$S4.3_1$ is characterized by the class of frames $\langle W, R\rangle$ in which R is reflexive, transitive, convergent, and such that, for all $x, y, z, w \in W$,*

$$\underline{C}.\ (xRy \wedge xRz) \supset (yRz \vee ((yRw \wedge y \neq w) \supset zRw))$$

PROOF. (a) For *soundness* it is sufficient to show that Lem_1 cannot be falsified in any model in which R satisfies $\underline{C}$. To show this, suppose that Lem_1 is false at x in some such model. Then there must be points y and z such that xRy and xRz and such that (i) $Lp \supset (q \supset Lq)$ is false at y and (ii) $Lq \supset p$ is false at z. From (i) it follows that (iii) Lp is true at y, (iv) q is true at y, and (v) Lq is false at y. By (iv) and (v) there must be some point w such that yRw and (vi) q is false at w; and hence (vii) $y \neq w$. Moreover from (ii) it follows that (viii) Lq is true at z and (ix) p is false at z. But now it is clear that $\underline{C}$ cannot be satisfied: for since we have xRy and xRz, to satisfy $\underline{C}$ we should have to have either yRz, which is impossible by (iii) and (ix), or else zRw, which is impossible by (viii) and (vi).

(b) For *Completeness* we use the method of canonical models. Since $S4.3_1$ is an extension of $S4.2$ it is sufficient to show that in the canonical model for $S4.3_1$, R satisfies $\underline{C}$. We first note that a straightforward transform of Lem_1 is

(1) $M(Lp \wedge q \wedge M \sim q) \supset L(\sim p \supset M \sim q)$

and that from (1) by $[q\vee \sim r/q]$ and PC we obtain

(2) $M(Lp \wedge (q\vee \sim r) \wedge M(\sim q \wedge r)) \supset L(\sim p \supset M(\sim q \wedge r))$.

Now let x, y, z, w be any points in the canonical model for $S4.3_1$ such that (i) xRy, (ii) xRz, (iii) $\sim yRz$, (iv) yRw, and (v) $y \neq w$. It will be sufficient to show that in that case we have zRw.

By (iii) there is some wff $I\alpha \in y$ such that (vi) $\sim \alpha \in z$. By (v) there is some $\beta \in y$ such that $\sim \beta \in w$. Let γ be any arbitrary wff in w. To show that zRw it is sufficient to show that $M\gamma \in z$.

Now since $\sim \beta \in w$ and $\gamma \in w$, then by (iv) we have $M(\sim \beta \wedge \gamma) \in y$. Since we also have $I\alpha \in y$ and $\beta \in y$ (and hence $\beta \vee \sim \gamma \in y$), then (i) we have $M(L\alpha \wedge (\beta \vee \sim \gamma) \wedge M(\sim \beta \wedge \gamma)) \in x$. Therefore by (2) we have $L(\sim \alpha \supset M(\sim \beta \wedge \gamma)) \in x$. Hence by (ii), $\sim \alpha \supset M(\sim \beta \wedge \gamma) \in z$. From this and (vi) we have $M(\sim \beta \wedge \gamma) \in z$, and therefore $My \in z$, which is what we required.

IV

The generalizations of Theorems 1-4 are as follows.

THEOREM 5. *Each* $S4.3_n$ *contains* $S4.3_{n+1}$.

SKETCH PROOF. For $n \geq 1$, to obtain Lem_{n+1} from Lem_n the key substitutions are $[p_2 \vee (p_1 \supset Lp_1)/p_1, p_3/p_2, \ldots, p_{n+1}/p_n]$. The required simplifications are then straightforward.

THEOREM 6. $S4.2$ *does not contain any* $S4.3_n$*; and if* $m > n$*,* $S4.3_m$ *does not contain* $S4.3_n$.

PROOF. The frame of Figure 3, with R assumed reflexive and transitive, is a frame for $S4.3_m$ (and of course for $S4.2$); but if $m > n$, Lem_n can be falsified at x.

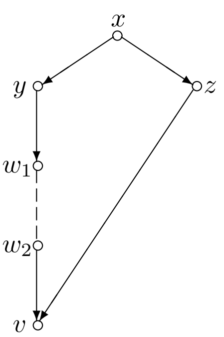

Fig. 3

Given R, let us say that $xR'y$ iff $xRy \wedge x \neq y$. And in general, let us say that $xR'^n y$ iff there are $z_0, \ldots, z_n$ such that (a) $z_0 = x$ and $z_n = y$, (b) $z_0 R z_1, \ldots, z_{n-1} R z_n$, and (c) for every i $(0 \leq i < n)$, $z_i \neq z_{i+1}$. (Less formally, we say that $xR'^n y$ iff y can be reached from x in n steps, each of which takes us from one element to a distinct one.) We interpret $xR'^0 y$ as $x = y$. We can then state

THEOREM 7. *Each $S4.3_n$ is characterized by the class of frames $\langle W, R \rangle$ in which R is reflexive, transitive, convergent, and such that, for all $x, y, z, w \in W$,*

$$(xRy \wedge xRz) \supset (yRz \vee (yR'^n w \supset zRw)).$$

The proof is a generalization of the proof of Theorem 4.

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