Bulletin of the **S**ection of **L**ogic
Volume 9/2 (1980), pp. 78–83
reedition 2010 [original edition, pp. 78–84]

G. E. Hughes

SOME EXTENSIONS OF THE BROUWERIAN LOGIC

The Brouwerian logic (B) is obtained by adding $Lp \supset p$ and $p \supset LMp$ to the minimal normal logic, K. $B + Lp \supset LLp$ yields $S5$, and $B + p \supset Lp$ the 'trivial' modal logic (Triv). All the above wff are of the form $Xp \supset Yp$, where X and Y are affirmative modalities (possibly empty). This paper discusses various other logics obtained by adding such wff to B.

By 'modality' we mean 'affirmative modality' throughout. We use X, Y, Z to range over modalities, and α, β over wff. By X' we mean the result of replacing L by M and M by L everywhere in X. L^n and M^n designate sequences of n L's and n M's respectively. If X consists of precisely k operators it is said to be of *degree* $\underline{k}$.

The following rules and theorem schemata (where α and β are any wff and X and Y are any modalities) hold in all normal extensions of B:

$R1.$	$\vdash \alpha \rightarrow \vdash X\alpha$	
$R2.$	$\vdash \alpha \supset \beta \rightarrow \vdash X\alpha \supset X\beta$	
$R3.$	$\vdash Xp \supset Yp \rightarrow \vdash Y'p \supset X'p$	
$T1.$	$\vdash X\alpha \supset XL^nM^n\alpha$	$(n \geq 0)$
$T2.$	$\vdash XM^nL^n\alpha \supset X\alpha$	$(n \geq 0)$
$T3.$	$\vdash L^{n+m}\alpha \supset L^n\alpha$	$(n, m \geq 0)$
$T4.$	$\vdash M^n\alpha \supset M^{n+m}\alpha$	$(n, m \geq 0)$
$T5.$	If X is of degree $k, \vdash X\alpha \supset M^k\alpha$	
$T6.$	If $n \geq m, \vdash (L^n\alpha \supset M^m\beta) \supset M^n(\alpha \supset \beta)$	
$T7.$	$\vdash M^n(\alpha \supset \beta) \equiv (L^n\alpha \supset M^n\beta)$	$(n \geq 0)$

A modality containing no sequence of the form $M^nL^nM^n$ or $L^nM^nL^n$ $(n \geq 1)$ will be said to be *in reduced form* (or simply, *reduced*). Since $\vdash_B M^nL^nM^n\alpha = M^n\alpha$ and $\vdash_B L^nM^nL^n\alpha = L^n\alpha$ $(n \geq 1)$, we can without loss of generality assume that all modalities are in reduced form.

The best-known extensions of B are the T_n^+ logics. For each $n \geq 0$, T_n^+ is defined as $B + T^n p \supset L^{n+1}p$. (Thus T_1^+ is $S5$ and T_0^+ is $Triv$). We note that in each T_n^+ the following theorem schema holds:

$T8.$ $\vdash X\alpha \supset YM^n\alpha$

A $wff\alpha$ will be said to be a B^* *formula* iff either $\vdash_B \alpha$ or $B + \alpha = T_n^+$ for some n.

LEMMA 1. *For every $n \geq 0$, if α is of the form $p \supset XL^{n+1}M^n p$ then $B + \alpha = T_n^+$.*

PROOF. By $T8$, T_n^+ contains $B + \alpha$. For the converse, assume B and $p \supset XL^{n+1}M^n p$. Then by $[L^n p/p]$ we have

(1)	$L^n p \supset XL^{n+1}M^n L^n p$	
(2)	$L^n p \supset XL^{n+1}p$	((1), $T2$)

Let k be the degree of X. Then

(3)	$L^n p \supset M^k L^{n+1}p$	((2), $T5$)
(4)	$L^k M^{n+1}p \supset M^n p$	((3), $R3$)

Since, by $R1$, $\vdash M^{n+1}\alpha \rightarrow \vdash L^k M^{n+1}\alpha$, (4) yields the rule: $\vdash M^{n+1}\alpha \rightarrow \vdash M^n\alpha$; and hence by repetition the rule:

(5)	$\vdash M^n\alpha \rightarrow \vdash M^n\alpha$	for every $m \geq n$
(6)	$L^{2n+k}p \supset M^k L^{2n+k+1}p$	((3)$[L^{n+k}p/p]$)
(7)	$M^{2n+k}(p \supset L^{2n+k+1}p)$	((6), $T6$)
(8)	$M^n(p \supset L^{2n+k+1}p)$	((7), (5))
(9)	$L^n p \supset M^n L^{2n+k+1}p$	((8), $T7$)
(10)	$\supset L^{n+k+1}p$	((9), $T2$)
(11)	$\supset L^{n+1}p$	((10), $T3$)

COROLLARY. *If Z ends with L, then $B + p \supset Zp = Triv$; and if Z ends with LIM, then $B + p \supset Zp = S5$.*

LEMMA 2. *If Z is a reduced modality not of the form $XL^{n+1}M^n$ then Z is of the form*

$$L^{a_1}M^{a_2} \ldots L^{a_{n-1}}M^{a_n}$$

where $a_1, \ldots, a_n$ are non-negative integers such that (i) $a_{n-1} \leq a_n$ and (ii) $a_1 < \ldots < a_{n-1}$.

Proof. Clearly every modality is of the form mentioned, where $a_1, \dots, a_n$ are *some* non-negative integers. The Lemma then follows from the easily verified facts (a) that if Z violates (i) then it is of the form $XL^{n+1}M^n$; and (b) that if it violates (ii) but not (i) then it contains some sequence $M^{a_j}L^{a_{j+1}}M^{a_{j+2}}$ or $L^{a_j}M^{a_{j+1}}L^{a_{j+2}}$ such that $a_j \geq a_{j+1} \leq a_{j+2}$; and hence is not in reduced form.

Lemma 3. *If Z is a reduced modality not of the form $XL^{n+1}M^n$ then $\vdash_B p \supset Zp$.*

The proof is straightforward from Lemma 2, $T1$ and $T4$.

Since, as noted above, every modality is equivalent in B to some reduced modality, Lemmas 1 and 3 immediately yield.

Theorem 1. *Every wff of the form $p \supset Zp$ is a B^* formula.*

Clearly, to show that α is a B^* formula it is sufficient to show that, given B, α is interdeducible with some wff of the form $p \supset Zp$.

Theorem 2. *Every wff of the form $M^jL^kp \supset Yp$ $(i, k \geq 0)$ is a B^* formula.*

Proof. Assume B and $M^jL^kp \supset Yp$. Then by $[M^kp/p]$ and $R2(X = L^j)$ we have $L^jM^jL^kM^kp \supset L^jYM^kp$. Whence by $T1$ we have $p \supset L^jYM^kp$, which is of the form $p \supset Zp$. Now assume B and $p \supset L^jYM^kp$. Then by $[L^kp/p]$ and $R2(X = M^j)$ we obtain $M^jL^kp \supset M^jL^jYM^kL^kp$. Whence by $T2$ we have $M^jL^kp \supset Yp$.

Theorem 3. *Every wff of the form $L^jM^kp \supset YM^lp$ $(j \geq k \leq l)$ is a B^* formula.*

Proof. Assume B and (1) $L^jM^kp \supset YM^lp$. Then since $j \geq k$, we have

(2) $L^jM^jp \supset YM^{l+j-k}p$ $\quad$ $((1)[M^{j-k}p/p])$

(3) $p \supset YM^{l+j-k}p$ $\quad$ $((2),\ T1)$

(3) is of the form $p \supset Zp$, and hence by Theorem 1 is a B^* formula. Now assume B and (3). Then

(4) $L^jM^kp \supset YM^{l+j-k}L^jM^kp$ $\quad$ $((3)[L^jM^kp/p])$

Since $k \leq l$, we have

(5)	$M^{l-k}M^j p \supset M^{l-k}p$	$(T2, R2\ (X = M^{l-k}))$
i.e.(6)	$M^{l+j-k}L^j p \supset M^{l-k}p$	
(7)	$YM^{l+j-k}L^j M^k p \supset YM^l p$	$((6)[M^k p/p], R2)$
(1)	$L^j M^k p \supset YM^l p$	$((4),\ (6),\ Syll)$

THEOREM 4. *If α is of the form $L^j M^k p \supset YL^{n+1}M^n p$ $(j \leq k)$ then $B + \alpha = T_n^+$ (and hence α is a B^* formula).*

PROOF. Assume B and (1) $L^j M^k p \supset YL^{n+1}M^n p$. From this and (1) we have $p \supset YL^{n+1}M^n p$; hence by Lemma 1, $B + (1)$ contains T_n^+. $T8$ shows the converse.

THEOREM 5. *If α is of the form $XMp \supset YLp$ then $B + \alpha = Triv$ (and hence, since $Triv = T_0^+$, α is a B^* formula).*

PROOF. Where X is any modality, the following are Theorems of B:

(1) $XM(p \supset Lp)$
(2) $XL(p \supset Lp) \supset X(Mp \supset LLp)$

Assume B and $XMp \supset YLp$. Then by $[p \supset Lp/p]$ and (1) we have

(3) $YL(p \supset Lp)$

Let Y be of degree k. If $k \leq 1$, (3) yields $p \supset Lp$ by $T2$ or $T3$. If $k > 1$, then

(4)	$M^k L(p \supset Lp)$	$((3),\ T5)$
(5)	$M^k(Mp \supset LLp)$	$((4),\ (2))$
(6)	$L^k Mp \supset M^k LLp$	$((5),\ T7)$
(7)	$M^k LL(p \supset Lp)$	$((6)[p \supset Lp/p],\ (1))$
(8)	$M^{k-1}L(p \supset Lp)$	$((7),\ T2)$

Whence by repetition we have $ML(p \supset Lp)$, and so by $T2$, $p \supset Lp$. Hence $B + XMp \supset YLp$ contains $Triv$; and $T8$ yields the converse.

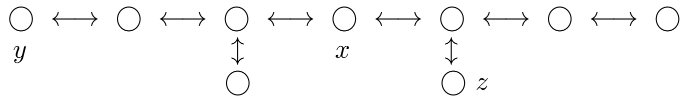
It might be thought that all wff of the form $Xp \supset Yp$ would turn out to be B^* formulae. This, however, is not the case.

THEOREM 6. *The logic $B + (C)$ $LMMp \supset MLLMMp$*

(a) is not contained in B, (b) is contained in T_2^+,
(b) does not contain any T_n^+.

PROOF. (b) holds by $T8$. For (a): every reflexive symmetrical frame is a frame for B, but (C) is false at x in the model on the (reflexive) frame of Fig. 1 in which $(V(p) = [y, z]$.

Fig. 1:



For (c), define a *chain* as a linear reflexive symmetrical frame, i.e. as (W, R) where $W = \{u_1, \ldots, u_j, \ldots\}$ and $u_j R u_k$ iff $k = j$ or $k = j + 1$ or $k = j - 1$. It is not hard to show that every chain is a frame for $B + (C)$. But in an $n + 2$-membered chain $L^n p \supset L^{n+1} p$ is false at u_1 if p is false at u_{n+2} and nowhere else.

THEOREM 7. *The logic* $B + (D) LLLMMp \supset MMMLLMp$

(a) is contained in $S5$, *(b) is independent of* T_2^+*;*
and (c) $T_2^+ + (D) = S5$.

PROOF. (a) holds by $T8$. For (c): $MMp \supset LLLMMp$ and $MMMLLMp \supset Mp$ are theorems of T_2^+. Hence $T_2^+ + (D)$ yields $MMp \supset Mp$, and therefore $S5$; and by (a) we have precisely $S5$. For (b): since T_2^+ is weaker than $S5$, (c) shows that (D) is not in T_2^+. Moreover (D) has the striking property of being valid on a 4-membered chain, thought not on one with 3 or more than 4 members. But as shown above, $LLp \supset LLLp$ is invalid on a 4-membered chain, and so $B + (D)$ does not contain T_2^+.

Note that (D), unlike (C), is valid on the frame of Fig. 1. Since (C), unlike (D), is valid on every chain, $B + (C)$ and $B + (D)$ are mutually independent.

Theorems 6 and 7 will generalize as follows: Let $(C_m) = L^{m-1} M^m p \supset M L^m M^m p$. Then if $m \geq 2$, $B + (C_m)$ is not contained in B, is contained in T_m^+, but does not contain any T_n^+. Let $(D_m) = L^{m+2} M^{m+1} p \supset M^{m+2} L^{m+1} M^m p$. Then if $m \geq 1$, $B + (D_m)$ is contained in T_m^+ but is independent of T_{m+1}^+, and $T_{m+1}^+ + (D_m) = T_m^+$. (Note: each (D_m) is valid on an $M + 3$-membered chain.)

The following questions suggest themselves: (1) What classes of frames, if any, characterize the $B + (C_m)$ and the $B + (D_m)$ logics? (2) Does any

wff of the form $Xp \supset Yp$, when added to B, give either a logic between $B + (D)$ and $S5$ or one independent of both T_2^+ and $B + (D)$?

Victoria University of Wellington
Wellington, New Zelland