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EQUIVALENTIAL LOGICS (I)

In the present note we continue the investigations undertaken in [2]. A full version of the paper has been submitted to Studia Logica.

§1

Our goal is to give a characterization of the so called factorial matrices for a logic. A matrix $\underline{M} = (\underline{A}, D)$ is *factorial* iff the greatest congruence $\Theta_{\underline{M}}$ of $\underline{M}$ coincides with the diagonal of A. Recall that Θ is a congruence of a matrix $\underline{M} = (\underline{A}, D)$ iff Θ is a congruence of the algebra $\underline{A}$ and for any $a, b \in A$, if $a\Theta b$ then

$$a \in D \ \text{ iff } \ b \in D.$$

If $\underline{M} = (\underline{A}, D)$ is a matrix for a propositional language $\underline{L}$, then $\Theta_{\underline{M}}$ can be characterized as follows: $a\Theta_{\underline{M}}b$ iff for each formula $\varphi \in L$, each propositional variable p occurring in φ and each homomorphism $h \in Hom(\underline{L}, \underline{A})$:

$$h(^a/_p(\varphi) \in D \ \text{ iff } \ h(^b/_p)(\varphi) \in D,$$

where $h(^a/_p) \in Hom(\underline{L}, \underline{A})$ is defined on the propositional variables of $\underline{L}$ as follows

$$h(^a/_p)(q) = \begin{cases} h(q) & \text{if } \ q \neq p \\ a & \text{if } \ q = p. \end{cases}$$

Thus for each matrix $\underline{M} = (\underline{A}, D)$, the quotient matrix $\underline{M}/\Theta_{\underline{M}} = (\underline{A}/\Theta_{\underline{M}}, D/\Theta_{\underline{M}})$ is factorial ($D/\Theta_{\underline{M}} = \{[a]_{\Theta_{\underline{M}}} : a \in D\}$). If $\mathbb{K}$ is a class of matrices for $\underline{L}$, then $\mathbb{K}^*$ denotes the class of factorial matrices $\{\underline{M}/\Theta_{\underline{M}} : \underline{M} \in \mathbb{K}\}$.

Our purpose is to investigate the following problem. Given a class $\mathbb{K}$ of matrices for a propositional language $\underline{L}$, let $C = Cn_{\mathbb{K}}$ be the consequence operation determined by $\mathbb{K}$ in $\underline{L}$. What model-theoretic operations should one impose on $\mathbb{K}^*$ in order to obtain the class $Matr(C)^*$ consisting of all factorial matrices from $Matr(C)$? We give a solution in the case of equivalential logics.

If $\alpha(p_1, \ldots, p_n)$ is a formula from L in n variables and $\underline{M} = (\underline{A}, D)$ is a matrix for $\underline{L}$, then $\alpha_{\underline{A}}$ (or $\alpha_{\underline{M}}$) denotes the corresponding n-ary function on A. If $a_1, \ldots, a_n \in A$, then $\alpha_{\underline{A}}[a_1/p_1, \ldots, a_n/p_n]$ (or simply $\alpha_{\underline{A}}[a_1, \ldots, a_n]$) denotes the value of $\alpha_{\underline{A}}$ in $\langle a_1, \ldots, a_n \rangle$.

By a *logic* we shall mean a pair $(\underline{L}, C)$, where $\underline{L}$ is a propositional language, i.e. a finitary absolutely free algebra generated by an infinite denumerable set of propositional variables, and C is a structural consequence operation on $\underline{L}$. Thus in this note we assume that propositional languages are *countable*.

Let $E(p, q)$ be a nonempty set of propositional formulas from $\underline{L}$ in two variables, p and q. We write $E(\alpha, \beta)$ to denote the set of all formulas which results by the simultaneous substitution of α for p and β for q in all formulas from E.

Definition 1. ([3]). A logic $(\underline{L}, C)$ is *equivalential* (with respect to a set $E(p, q)$) iff the following conditions hold true

(i) $E(\alpha, \alpha) \subseteq C(\emptyset)$
(ii) $E(\alpha, \beta) \subseteq C(E(\beta, \alpha))$
(iii) $E(\alpha, \gamma) \subseteq C(E(\alpha, \beta) \cup E(\beta, \gamma))$
(iv) for every n-ary connective F from $\underline{L}$ $(n < \omega)$
$E(F(\alpha_1 \ldots \alpha_n), F(\beta_1 \ldots \beta_n)) \subseteq C(E(\alpha_1, \beta_1) \cup \ldots \cup E(\alpha_n, \beta_n))$
(v) $\alpha \in C(E(\alpha, \beta) \cup \{\beta\})$.

If $(\underline{L}, C)$ is equivalential, then we refer to the above set $E(p, q)$ as to a set of C-equivalencies of the logic $(\underline{L}, C)$.

Examples (1). If $(\underline{L}, C)$ is equivalential with respect to a set $E(p, q)$ consisting of a single formula, then the unique element of E is usually denoted as $p \Leftrightarrow q$.

(2). A logic $(\underline{L}, C)$ is *implicational* ([4],[5]) iff among the connectives of $\underline{L}$ there exists a binary one, denoted $\Rightarrow$, such that for every formulas $\alpha, \beta, \gamma \in L$ and each variable p occurring in γ the following conditions are satisfied:

(i_1) $\alpha \Rightarrow \alpha \in C(\emptyset)$
(i_2) $\beta \in C(\alpha, \alpha \Rightarrow \beta)$

(i$_3$) $\alpha \Rightarrow \gamma \in C(\alpha \Rightarrow \beta, \beta \Rightarrow \gamma)$
(i$_4$) $\beta \Rightarrow \alpha \in C(\alpha)$
(i$_5$) $\gamma(\alpha/p) \Rightarrow \gamma(\beta/p) \in C(\alpha \Rightarrow \beta, \beta \Rightarrow \alpha)$

($\gamma(\alpha/p)$ results from γ by substitution of α for all occurrences of p.)

The connective $\Rightarrow$ mentioned above is called an *implication* of the logic $(\underline{L}, C)$.

Observe that if $(\underline{L}, C)$ is implicational and $\Rightarrow$ is its implicational, then $(\underline{L}, C)$ is equivalential with respect to the set $E(p,q) = \{p \Rightarrow q, q \Rightarrow p\}$, where $p \neq q$.

PROPOSITION 2. *If $(\underline{L}, C)$ is equivalential (w.r. to $E(p,q)$) and $\underline{M} = (\underline{A}, D) \in Matr(C)$, then $a\Theta_{\underline{M}}b$ iff for every $\gamma \in E$, $\gamma_{\underline{A}}[a,b] \in D$, where $\Theta_{\underline{M}}$ is the greatest congruence of $\underline{M}$.*

By the *Lindenbaum bundle* for a logic $(\underline{L}, C)$ we mean the family $\mathbb{L}_C$ of all matrices $(\underline{L}, X)$, where $X \in Th(C)$. ($Th(C) = \{X \subseteq L : C(X) = X\}$. It is easy to prove that a logic $(\underline{L}, C)$ is equivalent iff for each $X \in Th(C)$, the set $\{(\alpha, \beta) : E(\alpha, \beta) \subseteq X\}$ is the greatest congruence of the matrix $(\underline{L}, X)$.

THEOREM 3. (cf. Theorem 2 in [2]). *Let $\mathbb{K}$ be a class of matrices for $\underline{L}$. Assume that the logic $(\underline{L}, Cn_{\mathbb{K}})$ is equivalential w.r. to a set $E(p,q)$. The:*

(1) If $Cn_{\mathbb{K}}$ is finitistic and E is finite, then
$Matr(Cn_{\mathbb{K}})^* = SP_R(\mathbb{K}^*) = SPP_u(\mathbb{K}^*)$.
(2) If $Cn_{\mathbb{K}}$ is not finitistic, then
$Matr(Cn_{\mathbb{K}})^* = SP_{\sigma-R}(\mathbb{K}^*)$,

where $P_{\sigma-R}$ denotes the operation of taking σ-reduced products of matrices, i.e. reduced products modulo σ-filters.

COROLLARY 4. *Let $(\underline{L}, C)$ be equivalential with respect to a set $E(p,q)$. Then*

(1) If C is finitistic and E is finite, then
$Matr(C)^* = SP_R(\mathbb{L}_C^*) = SPP_u(\mathbb{L}_C^*)$.
(2) If C is not finitistic, then
$Matr(C)^* = SP_{\sigma-R}(\mathbb{L}_C^*)$.

where $\mathbb{L}_C^*$ is the factorial Lindenbaum bundle for C.

Addendum. In the note [2] we made use but we did not define the notion of a strong homomorphism of matrices. Let $\underline{M} = (\underline{A}, D)$, $\underline{N} = \underline{B}, E)$ be matrices for $\underline{L}$. A mapping $h : A \to B$ is said to be a *strong homomorphism* (or, equivalently – matrix homomorphism) iff h is a homomorphism from the algebra $\underline{A}$ into $\underline{B}$ and for every $a \in A$, $a \in D$ iff $h(a) \in E$. The above definition differs from the one accepted in model theory (cf. [1], p. 242).

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